

REGULARITY AND COMPACTNESS THEORY OF STABLE FREE-BOUNDARY MINIMAL HYPERSURFACES

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ABSTRACT. We study stable free-boundary minimal hypersurfaces in a complete $(n + 1)$ -dimensional Riemannian manifold-with-boundary $(M, \partial M)$, and establish the free-boundary version of the well-known (interior) Schoen–Simon regularity and compactness theory from [SS81]. Our approach follows the intrinsic PDE scheme introduced by the first-named author in [Bel25].

In the key step towards the compactness and regularity, we identify and address the two possible scenarios in which local weak flatness (of a stable free-boundary minimal hypersurface) is verified near a point on ∂M : first, weak flatness with respect to ∂M ; second, weak flatness with respect to a reference hypersurface-with-boundary that meets ∂M almost orthogonally. In both flatness regimes we prove sheeting theorems, i.e. a graphical decomposition (possibly in the multi-valued sense) and estimates for the stable minimal hypersurface.

The sheeting theorems are obtained, in their most basic analytic form, as ε -regularity theorems (where smallness of ε reflects regimes with weak flatness). The starting point is a careful identification of geometrically adapted tilt functions. We choose them so that they quantify flatness and keep track of the ambient geometry, and we show that they solve appropriate non-linear elliptic PDEs on the free-boundary minimal hypersurface. These PDEs are amenable to an intrinsic analysis (of the tilt functions) via iteration à la De Giorgi, using stability to obtain weak Caccioppoli-type inequalities that make the iteration viable, and leading to the ε -regularity.

The two ε -regularity results (and consequent sheeting theorems) are valid for immersed hypersurfaces that are allowed to have a small singular set (a priori of locally finite $(n - 2)$ -Hausdorff measure). When specialised to embeddings, they yield (in combination with soft classical tools) the compactness and regularity theorem: a sequence of stable free-boundary embedded minimal hypersurfaces with uniformly bounded area (possibly with a singular set of locally finite $(n - 2)$ -measure) admits a subsequence converging to a free-boundary minimal hypersurface, that is smoothly embedded away from a set of Hausdorff dimension at most $n - 7$ (its singular set). The convergence is smooth and graphical (possibly with multiplicity) away from this (singular) set.

As an application of the compactness and regularity, we extend the free-boundary Almgren–Pitts existence theory of [LZ21b] to arbitrary dimension. In every compact Riemannian manifold-with-boundary of dimension $n + 1$, there exists a free-boundary minimal hypersurface, smoothly embedded except possibly for a singular set of Hausdorff dimension at most $n - 7$.

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1. INTRODUCTION

Let $(M^{n+1}, \partial M, g)$ be a smooth Riemannian manifold-with-boundary. A smooth hypersurface-with-boundary $(\Sigma, \partial\Sigma) \subset (M, \partial M)$ is a *free-boundary minimal hypersurface* if its mean curvature vanishes and it meets ∂M orthogonally along $\partial\Sigma$. This characterises stationarity with respect to area for compactly supported deformations that preserve the constraint $\partial\Sigma \subset \partial M$. The purpose of this paper is to develop a regularity and compactness theory for stable free-boundary minimal hypersurfaces with a small singular set, and to apply it to the existence of optimally regular¹ free-boundary minimal hypersurfaces in compact Riemannian manifolds-with-boundary, in arbitrary dimension.

The analogous interior problem ($\partial M = \emptyset$) found its resolution in the Schoen–Simon theory, a cornerstone of the field. In [SS81], they proved that a stable minimal hypersurface (without boundary), smoothly and properly embedded outside a closed set of locally finite $\mathcal{H}^{n-2}$ -measure, extends to one that is smooth and embedded away from a singular set of Hausdorff dimension at most $n - 7$, and established compactness of this class in the varifold topology under uniform mass bounds. The main efforts in [SS81] are devoted to the proof of a sheeting theorem, which, combined with a slicing argument, establish the aforementioned compactness theorem. Finally, they used these results to obtain the existence of an optimally regular closed minimal hypersurface in every closed Riemannian manifold, extending the low-dimensional Almgren–Pitts theory [Pit81], which was based on the Schoen–Simon–Yau curvature estimates in [SSY75].

The Schoen–Simon theory also underpins Wickramasekera’s influential work [Wic14], which (among other merits) has been fundamental in the development of an Allen–Cahn min-max approach to the existence of closed minimal hypersurfaces in compact Riemannian manifolds [HT00, TW12, Gua18]. By analogy, a (future) free-boundary theory in the spirit of [Wic14] (i.e. for free-boundary stationary stable integral varifolds without classical singularities) would most likely have to rely on the sheeting results that we provide here. Such developments would further lead to the completion of an Allen–Cahn min-max approach to the existence of free-boundary minimal hypersurfaces, cf. [LPS26].

The low-dimensional Almgren–Pitts theory was recently extended to the free-boundary setting by the second-named author in collaboration with X. Zhou in [LZ21b]. The missing ingredient for the higher-dimensional theory was a comprehensive free-boundary analogue of Schoen–Simon’s compactness for stable hypersurfaces with singular set of locally finite $\mathcal{H}^{n-2}$ -measure. A partial adaptation of the techniques of [SS81] to the free-boundary setting was carried out in [GJ86b] for surfaces ($n = 2$) in Euclidean domains with strictly uniformly convex boundary. Subsequently, [DLR18] noted that the same arguments carry over in arbitrary dimension and for Riemannian domains with strictly uniformly convex boundaries. A noticeable effect of the convexity assumption in [GJ86b] and [DLR18] is that it prevents interior touching of Σ with ∂M (by interior touching we mean that $(\Sigma \setminus \partial\Sigma) \cap \partial M \neq \emptyset$). Our results confirm a posteriori that (in the convex case) touching is also ruled out under a limit procedure. Touching behaviour can on the other hand arise when ∂M is not convex, as easy examples show, and may happen in the limit even if we start from a sequence that presents no touching. Handling this possibility (especially in the limit) is crucial for the development of a compactness theory, as well as an existence theory. Indeed, [LZ21b] deal with this situation in low dimensions by means of curvature estimates [GLZ20] (in the spirit of Schoen–Simon–Yau [SSY75]), while the compactness theory developed here permits its handling in all dimensions.

¹i.e. smoothly embedded away from a codimension 7 singular set.

An intrinsic PDE approach to the Schoen–Simon theory was recently introduced by the first-named author in [Bel25], where the key ε -regularity result for the classical tilt function (in Euclidean space, with respect to a hyperplane) is obtained by direct (non-linear elliptic) PDE analysis, carried out on the hypersurface. Our proof follows this intrinsic paradigm: rather than linearising the minimal surface equation in an ambient coordinate system (as in [SS81, GJ86b]), we derive differential inequalities directly on the immersed hypersurface for geometrically adapted tilt functions. Our tilt functions retain information about the Riemannian geometry of M and of ∂M , which turns out to be crucial for the success of the method, and particularly useful at the boundary (see Section 1.2.1 below). The ε -regularity result is the key ingredient to obtain a sheeting theorem, from which the compactness theory will follow via soft classical geometric measure theory arguments.

Another advantage of the intrinsic PDE method lies in the fact that (unlike [SS81]) it allows to prove ε -regularity and sheeting theorems for stable *immersed* (rather than embedded) free-boundary minimal hypersurfaces with sufficiently small (non-immersed) singular set. Both sheeting theorems are new in the case of stable immersions, with the boundary-parallel one new even in the embedded setting.

1.1. Main result. Our main result is the following free-boundary counterpart of the Schoen–Simon compactness theory.

Theorem 1.1 (Compactness theory). *Let $n \geq 2$ and let $(M^{n+1}, \partial M, g)$ be a smooth complete Riemannian manifold-with-boundary. Consider a sequence of smooth (up to the boundary) embedded two-sided hypersurfaces-with-boundary $\{(\Sigma_k, \partial \Sigma_k)\}_k$. Assume the following, for each k .*

- (i) (a priori control on the singular set) $\mathcal{S}_k = \bar{\Sigma}_k \setminus \Sigma_k$ has locally finite $\mathcal{H}^{n-2}$ -measure, where the closure is taken in M , and the embedding is proper in $M \setminus \mathcal{S}_k$.
- (ii) (free boundary condition) $\partial \Sigma_k \subset \partial M$.
- (iii) (stationarity) Σ_k is minimal in its interior $\Sigma_k \setminus \partial \Sigma_k$, in the sense that its mean curvature vanishes; for every $p \in \partial \Sigma_k$ the tangent half-plane to Σ_k at p is orthogonal to $T_p \partial M$.
- (iv) (stability) for every $\varphi \in C_c^{0,1}(\Sigma_k)$ the stability inequality holds on Σ_k , namely

$$(1.1) \quad \int_{\Sigma_k} |\nabla_{\Sigma_k} \varphi|^2 - \int_{\Sigma_k} (|A_{\Sigma_k}|^2 + \text{Ric}_M(\nu_{\Sigma_k}, \nu_{\Sigma_k})) \varphi^2 - \int_{\partial \Sigma_k} \mathbb{I}_{\partial M}(\nu_{\Sigma_k}, \nu_{\Sigma_k}) \varphi^2 \geq 0,$$

where ν_{Σ_k} is a continuous determination of unit normal to Σ_k .

- (v) (mass bounds) For every compact set $K \subset M$ we have $\sup_k \mathcal{H}^n(\Sigma_k \cap K) < +\infty$.

Then Σ_k converges locally subsequentially in M to a smooth embedded hypersurface-with-boundary Σ such that $\mathcal{S} = \bar{\Sigma} \setminus \Sigma$ has $\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 7$; in addition, if $n = 7$ the set $\mathcal{S}$ is discrete. The convergence is locally smooth and graphical (possibly with multiplicity) on $M \setminus \mathcal{S}$. Moreover, the hypersurface Σ is a free-boundary minimal hypersurface (i.e. it satisfies (ii)-(iii) with Σ in place of Σ_k) and, when taken with multiplicity, Σ is the image of a two-sided stable free-boundary minimal immersion.

Remark 1.2 (Stability properties of the limit hypersurface Σ). More precisely, Σ in Theorem 1.1 (while not being necessarily two-sided) satisfies:

- If Γ is a two-sided connected component of Σ , with unit normal ν_Γ , then it satisfies the stability inequality (1.1) with Γ in place of Σ_k .
- If Γ is a one-sided connected component of Σ , then the inclusion $\Gamma \hookrightarrow M$ lifts to a two-to-one immersion $\tilde{\iota} : \tilde{\Gamma} \rightarrow M$ of the orientable double cover $\tilde{\Gamma}$ in the normal bundle $N\Gamma$, with image Γ . Then $\tilde{\Gamma}$ endowed with the pullback metric $\tilde{\iota}^*g$ has a globally-defined unit normal $\tilde{\nu}_{\tilde{\Gamma}}$ and

the following stability inequality holds

$$\int_{\tilde{\Gamma}} |\nabla_{\tilde{\Gamma}} \phi|^2 - \int_{\tilde{\Gamma}} (|A_{\tilde{\Gamma}}|^2 + \text{Ric}_M(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}})) \phi^2 - \int_{\partial \tilde{\Gamma}} \mathbb{I}_{\partial M}(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}}) \phi^2 \geq 0,$$

for all $\phi \in C_c^{0,1}(\tilde{\Gamma})$.

See Section 5.6 for more details.

Remark 1.3. Convergence aspects of Theorem 1.1 are naturally formulated in the language of *varifolds*. Indeed, under the assumptions of Theorem 1.1, each Σ_k defines a free-boundary stationary integral varifold $|\Sigma_k| := \mathbf{v}(\Sigma_k, \mathbb{I}_{\Sigma_k})$. (We show this in Appendix B, as a consequence of the size assumption on $\mathcal{S}_k$, see also Section 1.2.6.) We then prove that, up to a subsequence, the varifolds $|\Sigma_k|$ converge to the varifold $V = \theta|\Sigma|$, where Σ is as in Theorem 1.1, and θ is a locally constant integer-valued locally summable positive function on Σ . In particular, the density function θ is constant on each connected component of Σ . Moreover, if Γ is a one-sided connected component of Σ , then θ is necessarily an even positive integer on Γ . Finally, V is free-boundary stationary and stable, i.e. for every vector field² $X \in \mathfrak{X}^{\text{tan}}(M)$, we have $\delta V(X) = 0$ and $\delta^2 V(X) \geq 0$. See Section 5 for details.

Remark 1.4. We emphasise that, throughout this paper, every reference to *smoothness* or *smooth convergence* of hypersurfaces-with-boundary should be interpreted as *smoothness up to the boundary* and *smooth convergence up to the boundary* respectively.

Remark 1.5. Notice that the limit hypersurface Σ in Theorem 1.1 may still touch the boundary in its interior, namely it is possible that $(\Sigma \setminus \partial \Sigma) \cap \partial M \neq \emptyset$. However, since the convergence to Σ is smooth and graphical, the mean curvature of Σ vanishes on $(\Sigma \setminus \partial \Sigma) \cap \partial M$. This constrains the possible behaviours. For example, by the maximum principle, if ∂M is strictly mean-convex (namely $H_{\partial M} > 0$), then necessarily $(\Sigma \setminus \partial \Sigma) \cap \partial M = \emptyset$. Similarly, if $U \subset \partial M$ is a relatively open set and $U \subset (\Sigma \setminus \partial \Sigma) \cap \partial M$, then, again by the maximum principle, $H_{\partial M} \equiv 0$ on U .

It is also possible that $(\overline{\Sigma} \setminus (\partial \Sigma \cup \Sigma)) \cap \partial M \neq \emptyset$, namely that $\overline{\Sigma}$ touches the boundary at an interior singular point (see Example 1.16).

Remark 1.6. Notice that, as in the interior case, the dimension bound $\dim_{\mathcal{H}}(\mathcal{S}) \leq n - 7$ is sharp, as one can see by considering e.g. one half of the Simons cone, namely the area-minimising (and therefore stable) singular free-boundary minimal hypersurface $\Sigma^+ = \{\sum_{j=1}^4 x_j^2 = \sum_{j=5}^8 x_j^2; x_8 \geq 0\} \subset \mathbb{R}_+^8$, which can be obtained as a smooth (away from its singular set $\mathcal{S} = \{0\}$) limit of a sequence of stable smooth free-boundary minimal hypersurfaces Σ_k^+ . Indeed, it is well-known that the complement $\mathbb{R}^8 \setminus \Sigma$ of the full Simons' cone consists of two connected components E_+, E_- , and each of $E_{\pm}$ is foliated by rescalings $\{\lambda \Gamma_{\pm}\}_{\lambda > 0}$ of a smooth area-minimising hypersurface $\Gamma_{\pm}$ such that $\lambda \Gamma_{\pm} \rightarrow \Sigma$ as $\lambda \rightarrow 0^+$ both in the sense of varifolds, and locally smoothly away from the origin. Since each $\Gamma_{\pm}$ is smooth and invariant under reflection in $\{x_8 = 0\}$, then $\Sigma_k^+ := k^{-1}(\Gamma_+ \cap \{x_8 \geq 0\})$ is an area-minimising (hence stable) smooth free-boundary minimal hypersurface in $\mathbb{R}_+^8$ and $\Sigma_k^+ \rightarrow \Sigma^+$ (which has a codimension 7 singular set) as $k \rightarrow \infty$.

Remark 1.7. In dimensions $3 \leq n + 1 \leq 7$, where the singular set is empty, the compactness theory can equivalently be expressed as a curvature estimate. More precisely, our Theorem 1.1 implies [GLZ20, Theorem 1.1] under the additional assumption that the stable free boundary minimal hypersurface Σ is embedded. We note that the original theorem [GLZ20, Theorem 1.1] can now be strengthened, due to the first-named author's extension of the Schoen–Simon–Yau estimates

² $\mathfrak{X}^{\text{tan}}(M) = \{X \in C_c^1(M, TM) : X(x) \in T_x(\partial M) \text{ for all } x \in \partial M\}$.

to ambient dimension $n + 1 = 7$ and of the recent progress on the stable Bernstein problem. In particular, as a consequence of [Bel25, Theorem 2], the curvature estimate of [GLZ20, Theorem 1.1] holds for *immersed* stable free boundary minimal hypersurfaces in the full dimensional range $3 \leq n + 1 \leq 7$, and, as a consequence of [CL24, CLMS26, Maz24, HLW26] the constant C_1 in [GLZ20, Theorem 1.1] can be taken to be independent of the area bound C_0 .

As announced above, replacing the low-dimensional compactness input in [LZ21b] by Theorem 1.1 gives the following existence theorem, which is new for $n \geq 7$.

Theorem 1.8. *For any given compact Riemannian manifold-with-boundary $(M^{n+1}, \partial M, g)$ with $n \geq 2$, there exists a smooth embedded hypersurface-with-boundary Σ that satisfies the following properties.*

- $\partial \Sigma \subset \partial M$.
- Σ is minimal in its interior $\Sigma \setminus \partial \Sigma$ (in the sense that its mean curvature vanishes) and for every $p \in \partial \Sigma$ the tangent half-plane to Σ at p is orthogonal to $T_p \partial M$.
- $\bar{\Sigma} \setminus \Sigma$ has Hausdorff dimension at most $n - 7$.

Remark 1.9. Similarly to Remark 1.3, in the language of varifolds, the varifold $V = |\Sigma|$ induced by the free-boundary minimal hypersurface Σ in Theorem 1.8 is free-boundary stationary, namely its first variation δV vanishes along every vector field $X \in \mathfrak{X}^{\text{tan}}(M)$. Moreover, the singular set $\text{sing } V$ of V has Hausdorff dimension at most $n - 7$.

Remark 1.10. Note that in Theorem 1.8 the set $\text{spt } \|V\| \cap \partial M$ might be empty, i.e. V might be supported on an optimally regular closed minimal hypersurface contained in the interior of M (see Appendix D.1). However, under additional geometric assumptions on $(M^{n+1}, \partial M, g)$, for example if $\text{Ric}_g \geq 0$ on M and ∂M is strictly mean-convex, it is possible to guarantee that V is supported on an optimally regular free-boundary minimal hypersurface with non-empty boundary. See Appendix D.2 for a proof.

1.2. Ideas of the proof. Let us now explain the overall structure of the argument for Theorem 1.1, and the various intermediate results of independent interest we obtain en route to the compactness theorem. As we mentioned at the very beginning of this introductory section, the approach is based on the method developed in [Bel25], which relies on intrinsic PDE analysis on immersed minimal hypersurfaces. The proof of Theorem 1.1 is organised around two sheeting theorems (Theorem 1.12 and Theorem 1.15), which characterise stable singular free-boundary hypersurfaces that are L^2 -close to certain almost-flat models as unions of minimal graphs over these models. Although the compactness theorem (Theorem 1.1) only deals with stable *embedded* free-boundary minimal hypersurfaces (see Section 1.2.5 for more on this aspect), the sheeting results can be proven with barely any additional effort in the larger class of stable *immersed* free-boundary-minimal hypersurfaces (see Definition 2.6 for a precise definition of the class of hypersurfaces we consider).

Our first step in the free-boundary setting is to identify the two almost-flat models. In the first, we use as reference hypersurface a geodesic ball in the boundary ∂M (leading to the *boundary-parallel* sheeting result in Theorem 1.12); in the second, the reference hypersurface is a smooth embedded half-disc meeting ∂M orthogonally (leading to the *boundary-orthogonal* sheeting result in Theorem 1.15).

The analytic architecture is the same in the two cases: the sheeting theorem is obtained as a consequence of an ε -regularity theorem (Theorem 1.11 and Theorem 1.14 respectively) for an appropriately chosen scale-invariant tilt excess, which is supposed to measure closeness to the model. Once the correct geometrically adapted tilt function is identified for each case, the proof of the ε -regularity formally follows the same steps as in [Bel25]: first, a differential inequality is derived for

the tilt function; secondly, by using a truncation of the tilt function as a test function in the stability inequality, an intrinsic weak Caccioppoli inequality on Σ is obtained; finally, by using an appropriate Sobolev inequality to relate the L^2 energies of successive truncations on nested cylinders, an iteration à la De Giorgi can be set up to yield an L^2 -to- L^∞ estimate for the tilt function at a smaller scale. This L^2 -to- L^∞ estimate under the assumption of initial smallness is the essence of our ε -regularity. Once the tilt is uniformly small, projection onto the model is locally invertible; so properness and the smallness of the projected singular set promote local graphs to a finite multigraph structure.

In order to highlight the common abstract structure of the arguments, we will present the common ingredients (such as the Sobolev inequalities in Section 2.2.4 or the iteration lemmas in Section 2.3) as stand-alone results in Section 2.

1.2.1. Geometric choice of the tilt functions. Two considerations determine the comparison hypersurfaces and tilt functions used in this argument. The first concerns the metric in which the differential identities are derived. In their boundary curvature estimates, Grüter–Jost [GJ86b, Section 3] flatten the boundary by a local diffeomorphism and treat the transformed area integrand as a perturbation of Euclidean area, following the framework of Schoen–Simon [SS81]. In doing so, the exact variational information is inevitably lost: stationarity and stability condition become approximate conditions (for the Euclidean area), and the PDE for the tilt will feature differentiation of the Euclidean mean-curvature, a term that admits no natural estimate and is not amenable to the PDE methods we use. Instead, the tilt functions that we use here are naturally induced by the Riemannian metric, and this permits us to preserve all variational information.

The second consideration is compatibility with the boundary geometry, which affects the choices of both the *reference* for flatness, and the *tilt function* to measure flatness (with respect to said reference). This issue persists even when M is the closure of a smooth domain in $\mathbb{R}^{n+1}$ equipped with the Euclidean metric, and the n -area is already Euclidean. As the proof of the compactness and regularity theorem proceeds via tangent cone analysis, it might be very tempting to aim for sheeting theorems where flatness is captured by the standard Euclidean tilt function with respect to either the hyperplane $T_x\partial M \subset \mathbb{R}^{n+1}$, or a half-hyperplane orthogonal to $T_x\partial M$. In both cases, however, we would run into trouble with the PDE analysis for the tilt, because Euclidean tilt functions introduce terms arising from the free boundary that, in general, are not controlled by the estimates used in our iteration. We consequently adapt both the comparison hypersurface and the reference normal field to the curved boundary. In the boundary-parallel case, we compare with a local portion of ∂M itself and define the tilt g using its distance function. In the boundary-orthogonal case, we use a comparison half-disc meeting ∂M orthogonally and define the tilt τ using a smooth unit extension Z of its normal that is tangent to ∂M along the ambient boundary. These choices yield the boundary identities needed to eliminate or control the conormal terms in the Caccioppoli inequalities.

1.2.2. Boundary-parallel sheeting theorem. Let $d_{\partial M}$ be the distance to the boundary in a collar neighbourhood, and define

$$g := |\nabla_\Sigma d_{\partial M}|.$$

This will be our tilt function when the comparison hypersurface is chosen to be a geodesic ball in ∂M : it measures the tangential component of the boundary-normal direction, and hence the deviation of $T\Sigma$ from the tangent spaces to the parallel hypersurfaces of the collar. A crucial feature is that

$$g = 1 \quad \text{on } \partial\Sigma,$$

while it satisfies a PDE in Σ thanks to Bochner's formula. Thus an L^2 -to- L^∞ estimate which ensures $g < 1$ in a smaller domain automatically shows that there are *no* free-boundary points there. The subsequent sheeting is therefore by interior minimal graphs over ∂M . In particular,

boundary-parallel L^2 closeness rules out a degeneration in which boundary components collapse into a boundary-parallel limit through flatter and flatter half-catenoidal necks.

Theorem 1.11 (Boundary-parallel ε -regularity for the tilt). *Let M be as above, $p \in \partial M$, and let P be a geodesic ball in ∂M centred at p . Assume $\rho_0 > 0$ is such that P has a ρ_0 -collar neighbourhood $\mathcal{N}_{\rho_0}^+(P)$ in M , and*

$$(1.2) \quad |\nabla_g^2 d_{\partial M}|^2 + |\nabla_g^3 d_{\partial M}| + |\text{Rm}_g| \leq \Lambda^2 \quad \text{in } \mathcal{N}_{\rho_0}^+(P),$$

where $d_{\partial M}$ denotes the distance function from ∂M . There exist $\varepsilon_0 > 0, R_0 > 0$ depending only on the dimension n and the geometric constant Λ such that the following holds.

Let $\delta \in (0, \rho_0/2)$, $R \in (0, R_0)$, $\sigma \in (0, 1)$, and let $P_R = B_R^{\partial M}(p) \subset P$. Let Σ be a stable singular properly immersed free-boundary minimal hypersurface in $\mathcal{N}_{\rho_0}^+(P_R)$ such that $\bar{\Sigma} \subset \mathcal{N}_{\delta}^+(P_R)$ and $\partial\Sigma \subset P_R$. If

$$E_{\Sigma, P}(R)^2 := R^{-n} \int_{\Sigma} |\nabla_{\Sigma} d_{\partial M}|^2 < (1 - \sigma)^n \varepsilon^2 \quad \text{for some } \varepsilon < \varepsilon_0,$$

then

$$|\nabla_{\Sigma} d_{\partial M}| < C_0(1 - \sigma)^{-\frac{n}{n+2}} E_{\Sigma, P}(R)^{\frac{2}{n+2}}, \quad \text{on } \Sigma \cap \mathcal{N}_{\rho_0}^+(P_{\sigma R}),$$

where the constant $C_0 = C_0(n, \Lambda)$ only depends on the dimension n and the constant Λ .

Theorem 1.12 (Boundary-parallel sheeting). *Under the assumptions of Theorem 1.11, we further conclude that $\partial\Sigma \cap P_{R/2} = \emptyset$, and that there exist an integer $Q \geq 1$ and ordered Lipschitz functions*

$$0 \leq u_1 \leq u_2 \leq \dots \leq u_Q, \quad u_j: P_{R/2} \rightarrow [0, \delta),$$

such that

$$\bar{\Sigma} \cap \mathcal{N}_{\rho_0}^+(P_{R/2}) = \bigcup_{j=1}^Q F_{\partial M}(\text{graph}(u_j)),$$

where $F_{\partial M}(\text{graph}(u_j)) := \{F_{\partial M}(y, u_j(y)) : y \in P_{R/2}\}$ and $F_{\partial M}: P \times [0, \rho_0) \rightarrow M$ is the normal exponential map of ∂M (as defined in Section 2) restricted to P . Furthermore,

$$\text{Lip}_P(u_j) \leq C(n, \Lambda) E_{\Sigma, P}(R)^{\frac{2}{n+2}},$$

for every $j = 1, \dots, Q$, where the constant $C(n, \Lambda)$ only depends on the dimension n and the geometric constant Λ in (1.2). Moreover,

- (i) If we further assume $\mathcal{H}^{n-2}(\mathcal{S}) = 0$, then there exist smooth functions $v_1, v_2, \dots, v_Q: P_{R/2} \rightarrow [0, \delta)$ such that

$$\bar{\Sigma} \cap \mathcal{N}_{\rho_0}^+(P_{R/2}) = \bigcup_{j=1}^Q F_{\partial M}(\text{graph}(v_j)),$$

and with

$$\|\nabla^2 v_j\|_{C^0(P_{R/2})} + R^{\alpha} [\nabla^2 v_j]_{\alpha, P_{R/2}} \leq C(n, \alpha, \Lambda) \left(R^{-1} E_{\Sigma, P}(R)^{\frac{2}{n+2}} + 1 \right)$$

for all $j = 1, 2, \dots, Q$ and $\alpha \in (0, 1)$. In particular, $R|A_{\Sigma}| \leq C(n, \Lambda) \left(E_{\Sigma, P}(R)^{\frac{2}{n+2}} + R \right)$ on $\Sigma \cap \mathcal{N}_{\rho_0}^+(P_{R/2})$.

- (ii) If, on the other hand, we assume that Σ is embedded, then $\bar{\Sigma} \cap \mathcal{N}_{\rho_0}^+(P_{R/2}) = \Sigma \cap \mathcal{N}_{\rho_0}^+(P_{R/2})$ and $0 \leq u_1 < u_2 < \dots < u_Q$ are smooth with estimates

$$\|\nabla^2 u_j\|_{C^0(P_{R/2})} + R^{\alpha} [\nabla^2 u_j]_{\alpha, P_{R/2}} \leq C(n, \alpha, \Lambda) \left(R^{-1} E_{\Sigma, P}(R)^{\frac{2}{n+2}} + 1 \right)$$

for all $j = 1, 2, \dots, Q$ and $\alpha \in (0, 1)$. Lastly, $R|A_\Sigma| \leq C(n, \Lambda) \left(E_{\Sigma, P}(R)^{\frac{2}{n+2}} + R \right)$ on $\Sigma \cap \mathcal{N}_{\rho_0}^+(P_{R/2})$.

1.2.3. Boundary-orthogonal sheeting theorem. The second model is a smooth two-sided n -dimensional half-disc P meeting ∂M orthogonally. We need to define a tilt function that quantifies deviation from this model. Directly extending the unit normal to P as gradient of d_P is not suitable for the boundary calculation: the normal derivative of the associated tilt would contain the uncontrolled component $A(\eta, \eta)$, where η is the free-boundary conormal. We instead construct a unit vector field Z on a neighbourhood of P in M such that

$$Z|_P = \nu_P, \quad Z \in T(\partial M) \quad \text{along } \partial M.$$

Its flow preserves ∂M , and we measure tilt by

$$\tau := |Z^T|,$$

where Z^T is the projection of Z onto $T\Sigma$. Tangentiality of Z converts the boundary derivative of τ into terms involving only the second fundamental form of ∂M and derivatives of Z (see Lemma 4.9), so the boundary contribution can be handled by the trace inequality.

Remark 1.13. Our boundary-orthogonal sheeting theorem includes, as a special case, the result in [GJ86b, Section 3]. Our construction of Z replaces, in some sense, the flattening of ∂M performed in [GJ86b]. While M is therein locally described in coordinates as a half ball $B_+^{n+1}(0)$, with ∂M identified with $B^n(0) \times \{0\}$, we use (flow lines of) Z and $\nabla^g d_{\partial M}$ to reach a neighbourhood of P in M (adapted neighbourhood, using our terminology in Section 4.4), noting that Z preserves ∂M and $\nabla^g d_{\partial M}$ is orthogonal to it. Crucially, we keep the ambient metric and the variational information, while [GJ86b] work with the Euclidean metric in $B_+^{n+1}(0)$ (following [SS81]). That choice would not work for our method, since the Euclidean mean curvature of the transformed hypersurface becomes nonzero and its gradient inevitably appears in the estimates, producing error terms that are not controlled by the available hypotheses. On the other hand, the tilt function τ , being built from Z , reflects the Riemannian geometry of the ambient boundary; moreover, having left ambient metric and variational hypotheses unaltered, τ satisfies a PDE that allows us to successfully carry out the approach of [Bel25] in our setting. We note that Z and $\nabla^g d_{\partial M}$ play a similar role, in flavour, to the construction of a so-called orthogonal foliation in [LZ21a], used in a context of maximum principle for free-boundary minimal submanifolds.

More precisely, let P be a smooth two-sided embedded n -dimensional half-disc, let $\partial_0 P = P \cap \partial M$, and assume that P meets ∂M orthogonally along $\partial_0 P$. Such a P can be constructed locally, for instance taking the half-disk in Fermi coordinates centred at $p \in \partial M$. See Section 4.1 for further details. If $p \in \partial_0 P$, write $P_R = B_R^P(p)$. The adapted field Z has a flow map F_Z , and we set

$$N_\delta^Z(P_R) := F_Z(P_R \times (-\delta, \delta)).$$

Theorem 1.14 (Boundary-orthogonal ε -regularity for the tilt). *Let M be as above and let P be a smooth two-sided, embedded, n -dimensional half-disc in M meeting ∂M orthogonally along $\partial_0 P$. Let the vector field Z be as above, and let $p \in \partial_0 P = P \cap \partial M$. Assume $\rho_1 > 0$ is such that the flow map F_Z is a diffeomorphism $P_{\rho_1} \times (-\rho_1, \rho_1)$ onto the relatively open neighbourhood $\mathcal{N}_{\rho_1}^Z(P)$ of P in M . Assume that*

$$(1.3) \quad \begin{aligned} & \sup_{P_{\rho_1}} |A_P|^2 + \sup_{P_{\rho_1}} |\nabla_P A_P| + \sup_{P_{\rho_1}} |\nabla_P^2 A_P|^{2/3} \leq \Lambda^2, \\ & \sup |\nabla_g^2 d_{\partial M}|^2 + \sup |\nabla_g^3 d_{\partial M}| + \sup |\nabla_g^4 d_{\partial M}|^{2/3} \leq \Lambda^2, \\ & \sup |\text{Rm}_g| + \sup |\nabla_g \text{Rm}_g|^{2/3} + \sup |\nabla_g^2 \text{Rm}_g|^{1/2} \leq \Lambda^2, \end{aligned}$$

in $\mathcal{N}_{\rho_1}^Z(P)$, where $d_{\partial M}$ denotes the distance to ∂M in M , and A_P denotes the second fundamental form of P in M . Then there exist $\varepsilon_1 > 0$, $\delta_1 \in (0, \rho_1)$ and $R_1 > 0$ depending only on the dimension n and the geometric constant Λ such that the following holds.

Let $\delta \in (0, \delta_1)$, $R \in (0, R_1)$, $\sigma \in (0, 1)$. Assume Σ is a stable singular properly immersed free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset \mathcal{N}_{\delta}^Z(P_R)$ and $\partial\Sigma \subset (\partial M \cap \mathcal{N}_{\rho_1}^Z(P))$. If we let Z^T denote the orthogonal projection of Z onto $T\Sigma$, we have that if

$$E_{\Sigma, P}^Z(R)^2 := R^{-n} \int_{\Sigma} |Z^T|^2 < (1 - \sigma)^n \varepsilon^2 \quad \text{for some } \varepsilon < \varepsilon_1,$$

then

$$(1.4) \quad |Z^T| \leq C_1(n, \Lambda)(1 - \sigma)^{-\frac{n}{n+2}} E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \quad \text{on } \Sigma \cap \mathcal{N}_{\rho_1}^Z(P_{\sigma R}),$$

where the constant $C_1 = C_1(n, \Lambda)$ only depends on the dimension n and the constant Λ .

Theorem 1.15 (Boundary-orthogonal sheeting). *Under the assumptions of Theorem 1.14, there exist an integer $Q \geq 1$ and ordered Lipschitz functions*

$$u_1^Z \leq u_2^Z \leq \dots \leq u_Q^Z, \quad u_j^Z: P_{R/2} \rightarrow (-\delta, \delta),$$

such that

$$\bar{\Sigma} \cap \mathcal{N}_{\rho_1}^Z(P_{R/2}) = \bigcup_{j=1}^Q F_Z(\text{graph}(u_j^Z)),$$

where $F_Z(\text{graph}(u_j^Z)) = \{F_Z(y, u_j^Z(y)) : y \in P_{R/2}\}$ and $F_Z: P_{R/2} \times (-\delta, \delta) \rightarrow M$ is the flow map of the vector field Z . Furthermore,

$$(1.5) \quad \text{Lip}_P(u_j^Z) \leq C(n, \Lambda) \left(\delta + E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \right),$$

for every $j = 1, \dots, Q$, where the constant $C(n, \Lambda)$ only depends on the dimension n and the geometric constant Λ in (1.3). Moreover,

- (i) If we further assume $\mathcal{H}^{n-2}(\mathcal{S}) = 0$, then there exist smooth functions $v_1^Z, v_2^Z, \dots, v_Q^Z: P_{R/2} \rightarrow (-\delta, \delta)$ such that

$$\bar{\Sigma} \cap \mathcal{N}_{\rho_1}^Z(P_{R/2}) = \bigcup_{j=1}^Q F_Z(\text{graph}(v_j^Z)),$$

and with

$$\|\nabla^2 v_j^Z\|_{C^0(P_{R/2})} + R^\alpha [\nabla^2 v_j^Z]_{\alpha, P_{R/2}} \leq C(n, \alpha, \Lambda) \left(R^{-1} \left(E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} + \delta \right) + 1 \right)$$

for all $j = 1, 2, \dots, Q$ and $\alpha \in (0, 1)$. In particular,

$$R|A_\Sigma| \leq C(n, \Lambda) \left(\delta + R + E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \right)$$

on $\Sigma \cap \mathcal{N}_{\rho_1}^Z(P_{R/2})$.

- (ii) If, on the other hand, we assume that Σ is embedded, then $\bar{\Sigma} \cap \mathcal{N}_{\rho_1}^Z(P_{R/2}) = \Sigma \cap \mathcal{N}_{\rho_1}^Z(P_{R/2})$, $u_1^Z, \dots, u_Q^Z$ are smooth and

$$u_1^Z < u_2^Z < \dots < u_Q^Z \quad \text{on } P_{R/2}.$$

Moreover, the u_j^Z satisfy estimates

$$\|\nabla^2 u_j^Z\|_{C^0(P_{R/2})} + R^\alpha [\nabla^2 u_j^Z]_{\alpha, P_{R/2}} \leq C(n, \alpha, \Lambda) \left(R^{-1} \left(E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} + \delta \right) + 1 \right)$$

for all $j = 1, 2, \dots, Q$ and $\alpha \in (0, 1)$. Lastly, $R|A_\Sigma| \leq C(n, \Lambda) \left(\delta + R + E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \right)$ on $\Sigma \cap \mathcal{N}_{\rho_1}^Z(P_{R/2})$.

1.2.4. Examples. We now construct two examples showing that, in the immersed setting, when $\mathcal{H}^{n-2}(\mathcal{S}) > 0$ genuine singular points of branch type may be present both in $\bar{\Sigma} \setminus \bar{\partial\Sigma}$ (and may possibly lie on ∂M) and in $\bar{\partial\Sigma}$. By abuse of terminology, we will refer to these two situations as interior branch points of Σ , and boundary branch points of Σ , respectively. Neither stability nor the existence of a flat tangent cone implies regularity at a point of contact with ∂M . Example 1.16 exhibits an interior branch point lying on ∂M , at which the surface is tangent to ∂M and has a multiplicity-two planar tangent cone. The product construction in Example 1.17 produces a genuine free-boundary stable minimal immersion whose codimension-two branch set reaches the boundary and whose tangent cone there is a multiplicity-two half-hyperplane meeting ∂M orthogonally. Thus, both types of flat cone appearing in our boundary sheeting theory can occur at branch points and therefore cannot, for immersed hypersurfaces, yield regularity. On the other hand, the sheeting theorems imply the uniqueness of the flat tangent cone, as recorded in Corollary 1.18.

Example 1.16 (Interior branch point touching the boundary). We will now exhibit a branched minimal immersion with an interior branch point touching the boundary ∂M . Consider $x_3 = u_\pm(x)$, where $x = (x_1, x_2)$, satisfying $|u(x)| \leq C|x|^\alpha$. Let $b_R(x) = -R + \sqrt{R^2 - |x|^2} = -\frac{|x|^2}{2R} + O(|x|^4)$ be the upper half-sphere centred at $-Re_3$. If $\alpha > 2$, then

$$u_\pm(x) \geq -c|x|^\alpha > -\frac{|x|^2}{2R} + O(|x|^4) = b_R(x),$$

for $|x|$ sufficiently small. In particular, $u = \pm c\Re(z^{5/2})$ satisfies this property, but it is harmonic, not minimal. We will use the Weierstrass–Enneper parametrization to circumvent this. Consider $\Phi(\zeta) = (\zeta(1 - \zeta^6), i\zeta(1 + \zeta^6), 2\zeta^4)$, and note that $\Phi \cdot \Phi \equiv 0$. In particular,

$$X(\zeta) = \Re \int \Phi(\zeta) d\zeta = \Re \left(\frac{\zeta^2}{2} - \frac{\zeta^8}{8}, i\frac{\zeta^2}{2} + i\frac{\zeta^8}{8}, \frac{2\zeta^5}{5} \right) \in \mathbb{R}^3.$$

The Weierstrass–Enneper theorem (cf. [CM11, pag. 33]) shows that X is a branched minimal immersion, and that $\zeta = 0$ is an isolated branch point. Furthermore, writing $X = (X', X_3)$ with $X' \in \mathbb{R}^2$, we have

$$X'(\zeta) = \frac{1}{2}(\Re(\zeta^2), -\Im(\zeta^2)) + O(|\zeta|^8) \quad \text{and} \quad X_3(\zeta) = \frac{2}{5}\Re(\zeta^5),$$

so that $|X'(\zeta)| \simeq |\zeta|^2$, while $|X_3(\zeta)| \leq C|\zeta|^5 \leq C|X'(\zeta)|^{5/2}$, so that $X_3 \simeq \pm c\Re((x_1 + ix_2)^{5/2})$ as required. Now, for small enough ζ , there holds

$$|X(\zeta) + Re_3|^2 - R^2 = |X'(\zeta)|^2 + X_3(\zeta)^2 + 2RX_3(\zeta) \geq c|\zeta|^4 - C_R|\zeta|^5 > 0,$$

with equality holding at $\zeta = 0$. This is a smooth stable immersion $f : S := D \setminus \{0\} \rightarrow \mathbb{R}^3$, where D is a sufficiently small disc centred at the origin in $\mathbb{R}^2 = \mathbb{C}$. Stability follows from Barta's theorem (cf. [CM11, Lemma 1.36]). More precisely, if $\nu(\zeta)$ denotes the upward pointing unit normal, then $\langle \nu(\zeta), e_3 \rangle$ gives a positive Jacobi field. Take then as manifold-with-boundary $\mathbb{R}^3 \setminus B_R(-Re_3)$. We refer to [SW07] for more examples of stable branched minimal immersions.

Example 1.17 (Boundary branch point). Let $f : S \rightarrow \mathbb{R}^3$ be as in Example 1.16, so that in particular f is a two-sided stable branched minimal immersion, with $\partial\Sigma = \emptyset$ and discrete branch set $\mathcal{B}$. Set, $\Sigma = f(S)$, $N = \mathbb{R}^3$, and $F : S \times \mathbb{R}_+ \rightarrow \mathbb{R}_+^4$, as well as

$$\widehat{\Sigma} := \Sigma \times \mathbb{R}_+, \quad \widehat{M} := N \times \mathbb{R}_+ = \mathbb{R}_+^4, \quad F(x, y, s) := (f(x), s),$$

where $\mathbb{R}_+ = [0, \infty)$ and $\widehat{g} = g + ds^2$. Then $\partial\widehat{\Sigma} = \Sigma \times \{0\}$ and $F(\partial\widehat{\Sigma}) \subset \partial\widehat{M}$. For $X \in T_x\Sigma$, and $a \in \mathbb{R}$, $dF(X, a) = (df(X), a)$ so that $\text{rank } dF = \text{rank } df + 1$. Hence

$$\widehat{\Sigma}_{\text{reg}} = (\Sigma \setminus \mathcal{B}) \times \mathbb{R}_+, \quad \mathcal{B}_F = \mathcal{B} \times \mathbb{R}_+.$$

Thus $\dim \mathcal{B}_F = 1$, so $\mathcal{B}_F$ has codimension 2, and its boundary part is $\mathcal{B} \times \{0\}$. On the parameter manifold $S \times \mathbb{R}_+$, $F^*\widehat{g} = f^*g + ds^2$. If ν is a unit normal to f , then $\widehat{\nu} = (\nu, 0)$ is a unit normal to F , and the product connection gives $A_F((X, a), (Y, b)) = A_f(X, Y)$. All mixed and Euclidean-factor terms vanish; hence $\text{tr } A_F = \text{tr } A_f = 0$, so that F is minimal. Along $\partial\widehat{\Sigma}_{\text{reg}}$, the outward conormal of F and the outward normal to $\partial\widehat{M}$ are both $-\partial_s$. Thus F is free-boundary, see also Lemma C.2. Since $\partial\widehat{M}$ is totally geodesic, $|A_F|^2 = |A_f|^2$, and $\text{Ric}_{\widehat{M}}(\widehat{\nu}, \widehat{\nu}) = \text{Ric}_N(\nu, \nu)$. Therefore, for $\varphi \in C_c^\infty(\widehat{\Sigma}_{\text{reg}})$, writing $\varphi_s(x) := \varphi(x, s)$,

$$Q_F(\varphi) = \int_{\mathbb{R}_+} Q_f(\varphi_s) ds + \int_{\widehat{\Sigma}_{\text{reg}}} |\partial_s \varphi|^2 d\mu_F \geq 0.$$

By splitting a variation into odd and even components with respect to reflection in the totally geodesic boundary, we see that F is free-boundary stable. Stability extends across $\mathcal{B}_F$ by logarithmic cutoff. Lastly, if the tangent varifold of f at p is $Q|P|$, then that of F at $F(p, 0)$ is $Q|P \times \mathbb{R}_+|$, a multiplicity- Q half-hyperplane meeting $\partial\widehat{M}$ orthogonally.

Corollary 1.18 (Uniqueness of flat tangent cones). *Let $(\Sigma, \partial\Sigma)$ be as in Definition 2.12, and let $x \in \overline{\Sigma}$ be such that a tangent cone (to the stationary varifold $|\Sigma|$ at x) is either a half-hyperplane orthogonal to $T_x\partial M$, or the hyperplane $T_x\partial M$ (in both cases, a constant integer multiplicity may be present). Then the tangent cone is unique.*

1.2.5. From the sheeting theorems to the compactness theory. We finally indicate how the local estimates yield Theorem 1.1. The local mass bounds, together with the corresponding local first-variation bounds, give subsequential varifold compactness. In the present class, the required support convergence follows directly from a uniform lower density estimate obtained by combining the usual interior monotonicity formula with the boundary-centred monotonicity formula of [GLZ20]. In particular, no reference to the (computationally heavy) monotonicity of reflected density is needed.

Let V be a limit varifold and consider a tangent cone at a boundary point. After doubling in the tangent half-space, a cone with n -dimensional spine is flat and falls into one of the two models above: either a multiple of $T_x\partial M$, or a multiple of an orthogonal half-hyperplane. Varifold convergence gives height closeness to the relevant model. The height-to-tilt inequalities of Section 5.1 turn this into small L^2 tilt, and Theorems 1.12 and 1.15 then give smooth graphical convergence at a fixed scale. This proves regularity at every flat tangent cone and, at the same time, uniqueness of that cone.

Up to here, no distinction between embeddings and immersions is needed. The next step, on the other hand, uses embeddedness in an essential way. The Schoen–Simon slicing argument, adapted to the boundary, excludes stationary classical cones with an $(n - 1)$ -dimensional spine. For immersed hypersurfaces, sheets may branch or pass through one another, as Examples 1.16 and 1.17 illustrate, and this exclusion mechanism is no longer available in the same form (see Question 1.19 for recent progress).

Once the $(n - 1)$ -dimensional stratum is excluded, the standard dimension-reduction argument for stable cones gives $\dim_{\mathcal{H}} \text{sing } V \leq n - 7$, by adapting the usual argument to the case of free-boundary stationary varifolds. At this stage as well we do not need to introduce a reflected density or reflected monotonicity formula: the ordinary interior monotonicity formula and the boundary monotonicity formula of [GLZ20] suffice. Reflection is used only at the tangent-cone level, where it

becomes straightforward, as the manifold-with-boundary has become a half-space and the boundary has become a Euclidean hyperplane.

1.2.6. On the smallness assumption and the stationarity condition. Our assumptions on the singular set are also novel, and less restrictive than in other previous works on stable immersed free-boundary minimal hypersurfaces, such as [Wic08]. Indeed, we only require the set of non-immersed singular points to have locally finite $\mathcal{H}^{n-2}$ measure, and only impose stationarity away from this set. In order to be able to use the consequences of global stationarity, such as monotonicity of area ratios, we then show that stationarity extends across sets of locally finite $\mathcal{H}^{n-2}$ measure by constructing a function that blows up on the exceptional set and is superharmonic along every n -dimensional tangent plane (this part is inspired by a construction of Harvey–Lawson in [HL14]); a logarithmic Caccioppoli estimate then yields cutoff functions of vanishing tangential energy. See Appendix B for details.

1.3. Open questions and future directions. We finish this introduction by mentioning some questions and open problems that arose during the preparation of this work.

Question 1.19. Is there a compactness theory for stable singular properly immersed free-boundary minimal hypersurfaces with singular set of locally finite $\mathcal{H}^{n-2}$ -measure? This seems likely given the present work and the recent substantial advances in the interior theory in $\mathbb{R}^{n+1}$: in [WZ26a], Wang–Zhang produce a tilt function that allows them to extend the methods and conclusions of [Bel25], leading to a sharp $(n - 3)$ -dimensional bound for the non-branch singular set, and establish a branched sheeting theorem near cones that are unions of hyperplanes with spine of dimension $n - 1$ and $n - 2$. At the free boundary one must additionally understand boundary branch points and their interaction with the touching set.

Question 1.20. Can the boundary-orthogonal ε -regularity and sheeting theorems (Theorems 1.14 and 1.15) be proven under the weaker notion of *stability away from the touching set* used in [GWZ21, Definition 2.2]? The present proof uses the analytic stability inequality for all compactly supported scalar test functions, including across the touching set, and it is not immediate how to recover the same Caccioppoli estimate from the weaker hypothesis.

Notice that, on the other hand, the tilt function used in the boundary-parallel case naturally vanishes on the touching set, and it is an admissible test function for the stability inequality under the weaker notion of stability away from the touching set.

Question 1.21. Can the sheeting and compactness theorems above be generalised to the capillary setting? This appears to be promising, in light of the present work and of very recent or simultaneous progress on related aspects. A theory for stable minimal hypersurfaces-with-boundary with given capillary angle in a Euclidean half-space is developed in [WZ26b]. Moreover, in the forthcoming work [BJ26], a regularity and compactness theory is established, for stable hypersurfaces whose mean curvature is prescribed by an ambient function on a Riemannian manifold. A combination of the three works is likely to lead to a compactness and regularity theory for capillary hypersurfaces in the presence of a wetting energy and a potential energy.

Question 1.22. Let V be an n -dimensional rectifiable varifold which is stationary away from a relatively closed set S satisfying $\mathcal{H}^{n-1}(S) = 0$. Must V be stationary across S ? Proposition B.5 gives a positive answer when S has locally finite $\mathcal{H}^{n-2}$ -measure. Its proof is critical at that exponent and the construction used there no longer has the required properties under the weaker assumption $\mathcal{H}^{n-1}(S) = 0$.

Question 1.23. What structural hypothesis gives an optimal regularity and compactness theory for free-boundary stationary integral varifolds whose regular part is stable? The desired statement would be a free-boundary counterpart of Wickramasekera’s interior theory [Wic14], allowing stable regular parts without an a priori $\mathcal{H}^{n-2}$ bound on the singular set and excluding the appropriate boundary classical singularities. This question is the subject of ongoing work.

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2. PRELIMINARIES AND NOTATION

2.1. Inequalities and constants. We shall use the convention that $C(\alpha, \beta, \gamma, \dots)$ denotes a positive constant (which may change from line to line) and depends only on the parameters $\alpha, \beta, \gamma, \dots$ unless we specify otherwise.

2.2. Geometric setup. We refer to [Lee13, pp. 24–29] for the definition of (smooth) $(n + 1)$ -dimensional manifold-with-boundary M . We stress that ∂M is itself a (smooth) n -dimensional manifold, and that M contains it. The interior of M , i.e. $M \setminus \partial M$, is a (smooth) $(n + 1)$ -dimensional manifold³.

A Riemannian manifold-with-boundary (M, g) is obtained from M by endowing it with a smooth Riemannian metric g , with smoothness required up to the boundary⁴. Let ∇ denote the Levi-Civita connection on M induced by g . We will sometimes stress the dependence on the metric by writing ∇^g .

We denote by $\mathbf{n}$ the inward unit normal of ∂M , and we use the following notation for the (scalar-valued) second fundamental form of ∂M in M :

$$\mathbb{I}_{\partial M}(Y, Z) := g(\nabla_Y Z, \mathbf{n}) = -g(\nabla_Y \mathbf{n}, Z),$$

for vector fields Y and Z tangent to ∂M .

Remark 2.1. We note that, essentially by the collar neighbourhood construction ([Lee13, Theorems 9.25 and 9.26]), in the case $\partial M \neq \emptyset$, the manifold M can be realised as a proper (i.e. strict) closed subset $\iota(M) \subsetneq \Omega$ of a smooth $(n + 1)$ -manifold Ω , with $\iota: M \rightarrow \Omega$ a smooth proper embedding. In other words, M can be seen as a closed subset $M \subset \Omega$ with smooth boundary ∂M . Additionally, Ω can be taken to be smoothly homotopic to the interior of M . Furthermore, Ω admits a (smooth) Riemannian metric that restricts on M to the given one⁵.

Roughly speaking, the collar neighbourhood (which is a subset of M diffeomorphic to $\partial M \times [0, 1)$, with $\partial M \times \{0\}$ identified with ∂M) can be doubled, reflected and smoothly attached outside of M along ∂M , thus creating Ω .

³The notions of boundary and interior in this setting are not to be understood with respect to the topology of an ambient space in which we realise M as a subset.

⁴In local charts, this means that the metric is smooth in the interior and all of its derivatives extend continuously to the boundary.

⁵This follows essentially from a lemma of Borel, showing that a function is smooth up to the boundary, in the sense defined earlier, if and only if it admits a smooth extension across the boundary, see [Lee13, p.27].

By Remark 2.1 and [Lee18, Theorem 5.25 and Corollary 6.39(d)], whenever $P \subset \partial M$ is a relatively compact smooth submanifold of ∂M , then there exists some $\rho_P > 0$ such that the normal exponential map $F_{\partial M}$ of ∂M is a diffeomorphism of $P \times [0, \rho_P)$ onto its image $\mathcal{N}_{\rho_P}^+(P) := F_{\partial M}(P \times [0, \rho_P))$ of P in M .

2.2.1. Free-boundary minimal hypersurfaces. We will be interested in the following class of codimension one submanifolds-with-boundary, inside a smooth complete, Riemannian manifold-with-boundary (M^{n+1}, g) .

Definition 2.2 (Free-boundary minimal immersions and embeddings). Given a smooth n -dimensional manifold-with-boundary N and a relatively open set $U \subset M$, a smooth immersion $\iota : (N, \iota^*g) \rightarrow U \subset (M, g)$ is a *free-boundary minimal immersion* in U if

- (1) $\iota(\partial N) \subset \partial M \cap U$,
- (2) its mean curvature vanishes on the interior $N \setminus \partial N$ of N ,
- (3) for all $z \in \partial N$, the inward conormal of ∂N in N at z is mapped to $\mathbf{n}(\iota(z))$.

We shall denote the image $\iota(N)$ of ι by Σ , and let $\partial\Sigma := \iota(\partial N)$. Then $(\Sigma, \partial\Sigma)$ is said to be a smooth *immersed free-boundary minimal hypersurface* in $(M, \partial M)$. If, furthermore, ι is a homeomorphism of N onto its image, we will say that $(\Sigma, \partial\Sigma)$ is a smooth *embedded free-boundary hypersurface* in $(M, \partial M)$.

In what follows we shall often simplify the notation and write “ Σ is a smooth immersed free-boundary minimal hypersurface”, without explicitly referring to the manifold-with-boundary N or the map $\iota : N \rightarrow M$. Similarly, we shall often drop the more cumbersome notation $(M, \partial M, g)$, $(\Sigma, \partial\Sigma)$ for (Riemannian) manifolds-with-boundary and immersed/embedded hypersurfaces with boundary, in favour of the simpler (M, g) , Σ .

Remark 2.3. Note that the intersection of the interior of Σ and ∂M is not necessarily empty as easy examples show. We stress that the orthogonality between Σ and ∂M is only imposed at points in $\partial\Sigma$.

Remark 2.4. Notice that the conditions in Definition 2.2 imply that the immersion $\iota : N \rightarrow M$ is stationary for area with respect to compactly supported variations $(\iota_t : N \rightarrow M)_{t \in (-\delta, \delta)}$ which preserve the inclusion $\iota_t(\partial N) \subset \partial M$ for all t , as it is easily seen from the first variation formula for area⁶.

Notation 2.5. From now on, whenever for $A \subset M$ we write an integral $\int_{\Sigma \cap A} f$, we shall always implicitly assume it is shorthand for the integral $\int_{\iota^{-1}(A)} f \circ \iota d\mathcal{H}_{\iota^*g}^n$ over $\iota^{-1}(A)$ with respect to the induced metric ι^*g . Similarly quantities such as the second fundamental form A_Σ or a unit normal ν_Σ are to be understood implicitly as quantities on N .

2.2.2. Singular free-boundary minimal hypersurfaces. We will be concerned with the class of *stable* immersed free-boundary minimal hypersurfaces (see Definition 2.12 below). It is well-known that allowing a small singular set is unavoidable for a comprehensive theory in this class (e.g. for regularity, compactness, existence). We shall do so by introducing the notion of *singular properly immersed free-boundary minimal hypersurfaces*, where we impose some a priori size constraint on the singular set.

⁶Notice that the conditions in Definition 2.2 are stronger than stationarity with respect to compactly supported variations which map ∂N into the boundary ∂M . Indeed stationarity only implies the vanishing of the mean curvature of Σ away from the *touching set* $(\Sigma \setminus \partial\Sigma) \cap \partial M$, which is not assumed to be empty, as observed in Remark 2.3.

Definition 2.6 (Singular properly immersed free-boundary minimal hypersurfaces). Let $\iota : N \rightarrow M$ be a free-boundary minimal immersion as in Definition 2.2 with image $\Sigma = \iota(N)$, and let $\mathcal{S} = \overline{\Sigma} \setminus \Sigma$, where the closure is taken in M . We shall say that Σ (or, more precisely, $\overline{\Sigma}$) is a *singular properly immersed free-boundary minimal hypersurface* in M if:

- (1) The immersion ι is proper in $M \setminus \mathcal{S}$, i.e. for every compact set $K \subset M \setminus \mathcal{S}$ the inverse image $\iota^{-1}(K)$ is compact in N .
- (2) The measure $\mathcal{H}^n \llcorner \Sigma$ is locally finite, in the sense that for any compact $K \subset M$ we have $\mathcal{H}^n(\Sigma \cap K) < \infty$.
- (3) The set $\mathcal{S} := \overline{\Sigma} \setminus \Sigma$ has locally finite $\mathcal{H}^{n-2}$ -measure (in M), i.e. for any compact $K \subset M$ we have $\mathcal{H}^{n-2}(\mathcal{S} \cap K) < \infty$.

We shall refer to $\mathcal{S}$ as the *singular set* of Σ .

Notice that by (2), a singular properly immersed free-boundary minimal hypersurface Σ defines an integral n -varifold $V = |\Sigma|$ in M with weight measure $\|V\| = \mathcal{H}^n \llcorner \Sigma$ and $\text{spt } \|V\| = \overline{\Sigma}$. Moreover, by definition of a smooth free boundary minimal immersion, V is free-boundary stationary in $M \setminus \mathcal{S}$ in the sense of Appendix C.1.

Remark 2.7. Under the hypotheses of Definition 2.6 the varifold $V = |\Sigma|$ is free-boundary stationary in M in the sense of Section 5.2. More precisely, under those assumptions one can extend stationarity from $M \setminus \mathcal{S}$ to M (thus giving e.g. area-ratio estimates on balls centred on $\mathcal{S}$ by the monotonicity formula), see the discussion in Appendix B (Proposition B.5 in particular).

Remark 2.8. As stated, in Definition 2.6 we may have points in $\mathcal{S}$ that are not genuinely singular, e.g. we take Σ to be a punctured equatorial disc in $\mathbb{B}^3$ and $\mathcal{S}$ to be the missing point. Strictly speaking, one should define the singular set $\text{sing}(\Sigma)$ as the set of all $x \in M$ such that for every $r > 0$ the intersection of $\overline{\Sigma}$ with the geodesic ball $B_r(x)$ in Ω is not the image of a smooth free-boundary immersion in M which is *proper* in $B_r(x)$, that is, $\text{sing}(\Sigma)$ is the set of points across which Σ cannot be smoothly extended as a proper free-boundary immersion⁷. However, clearly $\text{sing}(\Sigma) \subset \mathcal{S}$ and, after extending Σ across its removable singularities, one may take $\mathcal{S} = \text{sing}(\Sigma)$.

Remark 2.9. We will speak of a smooth properly immersed free-boundary minimal hypersurface when $\mathcal{S} = \emptyset$. In this case the notion reduces to Definition 2.2 with the additional assumption of properness in M .

Remark 2.10. In our main compactness theorem, cf. Theorem 1.1, we specialise to (possibly) singular embeddings, in which case the immersion in Definition 2.6 is assumed to be an embedding. We also recall that a proper injective immersion with closed image is necessarily an embedding.

Remark 2.11. When we refer to a singular properly immersed free-boundary minimal hypersurface as being *two-sided* we shall always mean that there exists a continuous global determination of unit normal (more precisely, that the pullback normal bundle on N is trivial, and a global unit normal is a continuous section of the pullback unit normal bundle on N).

In what follows, we shall often simply write “ Σ is a singular free-boundary minimal hypersurface in M ”. This is always to be interpreted as shorthand for “ Σ is a singular properly immersed free-boundary minimal hypersurface in M ” in the sense of Definition 2.6, unless we specify otherwise.

⁷The set $\text{sing}(\Sigma)$ corresponds to the (*non-immersed*) *singular set* of the varifold $|\Sigma|$ induced by Σ , in the definition of [Wic08].

2.2.3. Stability. We shall take the following analytic inequality as our notion of stability for singular properly immersed free-boundary minimal hypersurfaces. We shall restrict ourselves to two-sided hypersurfaces, as we make clear in the following definition.

Definition 2.12 (Stable free-boundary minimal hypersurface). Let $\Sigma = \iota(N)$ be a singular properly immersed free-boundary minimal hypersurface in the sense of Definition 2.6. Further assume Σ to be two-sided, with unit normal ν_Σ and second fundamental form A_Σ ⁸. We say that Σ is *stable* if the following inequality holds

$$(2.1) \quad \int_N |\nabla_N \varphi|^2 - (|A_\Sigma|^2 + \text{Ric}_M(\nu_\Sigma, \nu_\Sigma))\varphi^2 - \int_{\partial N} \mathbb{I}_{\partial M}(\nu_\Sigma, \nu_\Sigma)\varphi^2 \geq 0$$

for all $\varphi \in C_c^{0,1}(N)$, where N is endowed with ι^*g .

Remark 2.13. It turns out (see Definition 2.6 and the discussion in Appendix B) that the assumptions in Definition 2.6 imply that the singular set $\mathcal{S}$ has *vanishing relative 2-capacity in Σ* , denoted $\text{cap}_2(\mathcal{S}) = 0$. Thus, (2.1) extends to any test function of the form $\varphi = \phi \circ \iota$ for $\phi \in C_c^1(M)$. The proof is by approximation using the cutoff functions provided by Corollary B.6 (see Lemma 2.14 for a similar argument).

From now on, following the convention laid out in notation 2.5, when we want to plug a test function of the form $\varphi = \phi \circ \iota$ for $\phi \in C_c^1(M)$ into (2.1), we shall always write integrals directly on Σ ; that is, we shall write $\int_\Sigma |\nabla_\Sigma \phi|^2$ in place of $\int_N |\nabla_N(\phi \circ \iota)|^2$.

2.2.4. Trace and Sobolev inequality. Fix a relatively compact open set $U \subset M$ such that the distance function $d_{\partial M}$ to ∂M is smooth on U . Let Σ be a two-sided singular free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset U$. Then, appropriate versions of the classical trace inequality and Sobolev inequality hold for Σ .

In this subsection, functions on Σ are understood on the parameter manifold N , as in Notation 2.5.

Lemma 2.14. *Let Σ be a two-sided singular free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset U$. Then if f is a non-negative compactly supported Lipschitz function in U , we have*

$$(2.2) \quad \int_{\partial \Sigma} f \leq \int_\Sigma |\nabla_\Sigma f| + \left(\sqrt{n} \sup_U |\nabla_g^2 d_{\partial M}| \right) \int_\Sigma f.$$

For simplicity, write $C_{\text{tr}} = C_{\text{tr}}(n, U) := \sqrt{n} \sup_U |\nabla_g^2 d_{\partial M}|$.

Proof. Let us first give a one-line proof in the case f is supported away from $\mathcal{S}$ (see also [Ede16, Lemma 2.1]), and then extend it to the more general case of f compactly supported in U by a (more or less standard) excision argument based on Corollary B.6.

Suppose f is supported away from $\mathcal{S}$, then by the divergence theorem on the immersed hypersurface Σ we have

$$\int_{\partial \Sigma} f = \int_{\partial \Sigma} g(f \nabla^g d_{\partial M}, \eta) = - \int_\Sigma \text{div}_\Sigma(f \nabla^g d_{\partial M}) = - \int_\Sigma g(\nabla_\Sigma f, \nabla^g d_{\partial M}) + f \text{tr}_\Sigma(\nabla_g^2 d_{\partial M}),$$

where in the last equality we used that the mean curvature of Σ vanishes identically. Equation (2.2) follows directly.

For a general f , let $K \Subset U$ be such that $\text{spt } f \subset K$. By Corollary B.6, there exist cutoffs $\chi_j \in C_c^{0,1}(U)$ such that $0 \leq \chi_j \leq 1$, $\text{spt } \chi_j \subset U \setminus \mathcal{S}$, $\chi_j \rightarrow 1$ locally on Σ , and $\int_\Sigma |\nabla_\Sigma \chi_j|^2 \rightarrow 0$. Apply (2.2) to $f_j := \chi_j f \circ \iota$ to get

$$\int_{\partial \Sigma} f_j \leq \int_\Sigma |\nabla_\Sigma(\chi_j f)| + C_{\text{tr}} \int_\Sigma f_j \leq \int_\Sigma (\chi_j |\nabla_\Sigma f| + f |\nabla_\Sigma \chi_j|) + C_{\text{tr}} \int_\Sigma f_j.$$

⁸More precisely, ν_Σ and A_Σ are tensor fields on N along ι , see notation 2.5.

Notice that by (ii) of Definition 2.6, Σ^9 has locally finite mass, so by the properties of the cutoffs χ_j we have

$$\int_{\Sigma} f |\nabla_{\Sigma} \chi_j| \leq \|f\|_{L^{\infty}} \mathcal{H}^n(\Sigma \cap K)^{1/2} \left(\int_{\Sigma \cap K} |\nabla \chi_j|^2 \right)^{1/2} \rightarrow 0.$$

The other terms on the right hand side converge by dominated convergence, and

$$\int_{\partial \Sigma} f \leq \liminf_j \int_{\partial \Sigma} f_j$$

by Fatou's lemma. This concludes the proof. $\square$

Remark 2.15. Notice that by taking $f \equiv 1$ in (2.2), we obtain an upper bound for $\mathcal{H}^{n-1}(\partial \Sigma)$ in terms of the geometry of U and $\mathcal{H}^n(\Sigma)$.

The trace inequality turns out to be very useful in dealing with *stable* singular free-boundary minimal hypersurfaces, because it allows to absorb the boundary term in the stability inequality into an interior term.

Lemma 2.16. *Let Σ be a stable singular free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset U$. For every $\alpha > 0$ there is a finite constant $C(n, \alpha, U)$ depending on $n, \alpha, \sup_U |\text{Ric}_g|, \sup_U |\nabla_g^2 d_{\partial M}|$ such that*

$$\int_{\Sigma} |A|^2 \zeta^2 \leq (1 + \alpha) \int_{\Sigma} |\nabla_{\Sigma} \zeta|^2 + C(n, \alpha, U) \int_{\Sigma} \zeta^2$$

for all bounded compactly supported Lipschitz functions ζ on U . More precisely, $C(n, \alpha, U)$ can be taken to be $\sup_U |\text{Ric}_g| + (4\alpha^{-1} + \sqrt{n}) \sup_U |\nabla_g^2 d_{\partial M}|^2$.

Proof. Let us first consider the case where ζ is supported away from $\mathcal{S}$. Then (2.1) holds. Therefore, by Lemma 2.14 and Young's inequality

$$\left| \int_{\partial \Sigma} \mathbb{I}_{\partial M}(\nu, \nu) \zeta^2 \right| \leq \alpha \int_{\Sigma} |\nabla_{\Sigma} \zeta|^2 + \|\mathbb{I}_{\partial M}\|_{L^{\infty}} (\alpha^{-1} \|\mathbb{I}_{\partial M}\|_{L^{\infty}} + C_{\text{tr}}) \int_{\Sigma} \zeta^2$$

This proves the result for compactly supported bounded Lipschitz functions ζ with $\text{spt } \zeta \subset U \setminus \mathcal{S}$.

For the general case, it is enough to apply the estimate we have just obtained to $\zeta_j := \chi_j \zeta \in C_c^{0,1}(U \setminus \mathcal{S})$, where χ_j is as in the proof of Lemma 2.14. $\square$

Finally, we are going to need to use the following Sobolev inequality, which follows from Edelen's Michael–Simon inequality for smooth free-boundary hypersurfaces in [Ede16, Theorem 2.2].

Lemma 2.17. *Let Σ be a two-sided singular free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset U$. There are a dimensional constant $C_{\text{Sob}} = C_{\text{Sob}}(n) > 0$ and a geometric constant $K_{\text{Sob}} = K_{\text{Sob}}(n, U)$ depending on n and $\sup_U |\nabla_g^2 d_{\partial M}| > 0$ such that for all compactly supported Lipschitz functions ζ on U , we have*

$$(2.3) \quad \left(\int_{\Sigma} |\zeta|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_{\text{Sob}} \left(\int_{\Sigma} |\nabla_{\Sigma} \zeta|^2 + K_{\text{Sob}}^2 \int_{\Sigma} \zeta^2 \right) \quad \text{for } n \geq 3,$$

and

$$(2.4) \quad \left(\int_{\Sigma} |\zeta|^4 \right)^{\frac{1}{2}} \leq C_{\text{Sob}} \left(\int_{\Sigma} \zeta^2 \right)^{\frac{1}{2}} \left(\int_{\Sigma} |\nabla_{\Sigma} \zeta|^2 + K_{\text{Sob}}^2 \int_{\Sigma} \zeta^2 \right)^{\frac{1}{2}} \quad \text{for } n = 2.$$

⁹or, more precisely, the measure $\mathcal{H}^n \llcorner \Sigma$.

Proof. For ζ supported away from $\mathcal{S}$, this follows from the free-boundary Michael-Simon inequality in [Ede16, Theorem 2.2] and the trace inequality (2.2) applied to $f = |\zeta|^{\frac{2(n-1)}{n-2}}$ for $n \geq 3$ and $f = \zeta^2$ for $n = 2$, followed by Young's inequality.

The general case then follows from the same approximation argument as in Lemma 2.16 using Corollary B.6. $\square$

2.3. Iteration à la De Giorgi. Now let Σ be a free-boundary minimal hypersurface in M . In what follows we shall be interested in the situation where Σ is weakly close, in a sense to be made precise later, to a smooth embedded two-sided hypersurface $P \subset M$ of one of the following two types:

- P is a geodesic ball $B_R^{\partial M}(p)$ in ∂M centred at a point $p \in \partial M$ and $0 < R < \text{inj}_p(\partial M)$ (see Section 3);
- P is an n -dimensional free-boundary half-disc in M intersecting ∂M orthogonally (see Section 4).

As explained in the Introduction, the central idea in the proofs of both Theorem 1.11 and Theorem 1.14 is an L^2 -to- L^∞ estimate for the tilt relative to P , which is obtained via a non-linear adaptation of De Giorgi's iteration technique following [Bel25].

The overall strategy in both cases is very similar: first, we introduce a *tilt function* to measure the deviation of the normal of Σ from the normal of P , and show that this function satisfies a differential inequality involving the norm $|A|^2$ of the second fundamental form of Σ ; then by using truncations of this tilt as appropriate test functions in the stability inequality, we obtain an intrinsic Caccioppoli-type inequality on Σ ; finally, thanks to the differential inequality satisfied by the tilt and the Sobolev inequality in Section 2.2.4, we set up a De Giorgi-style iterative scheme which results in a nonlinear recurrence equation for the L^2 -norm of successive truncations of the tilt function on smaller and smaller cylinders. The superlinear nature of the recurrence (see e.g. [Bel25, Lemma B.1]) implies that, provided the L^2 -norm of the tilt is sufficiently small at the original scale, then the L^2 -norm of successive truncations on shrinking cylinders must go to 0, thereby providing an L^∞ estimate for the tilt function at a smaller scale.

We now formulate this procedure abstractly, in a framework that applies to both tilt functions we shall use in Section 3 and Section 4.

2.3.1. Setup. Let Σ be a two-sided singular free-boundary minimal hypersurface in M and assume $\bar{\Sigma} \subset U$, where U is a relatively compact open subset of M . Let $\sigma \in (0, 1)$, $R > 0$ and let $\{U_r\}_{0 < r \leq R}$ be a nested family (increasing in r) of relatively open subsets of Σ . Assume that for all $\sigma R \leq r < r' \leq R$, there exists a compactly supported Lipschitz function $\phi_{r,r'}$ on Σ such that

$$0 \leq \phi_{r,r'} \leq 1, \quad \phi_{r,r'} \equiv 1 \text{ on } U_r, \quad \text{spt } \phi_{r,r'} \subset U_{r'}, \quad |\nabla_\Sigma \phi_{r,r'}| \leq \frac{C_c}{r' - r}$$

for some constant C_c independent of r, r' .

Let $u : \Sigma \rightarrow [0, 1]$ be a locally Lipschitz function on Σ and assume that there are constants $k_* \in (0, 1]$, $C_{\text{Cac}}, K_{\text{Cac}} > 0$ such that for every $k \in [0, k_*]$ and every compactly supported Lipschitz function $\phi \in C_c^{0,1}(\Sigma)$, the following Caccioppoli-type inequality is satisfied

$$(2.5) \quad \int_{\{u > k\}} \left(1 - \frac{k}{u}\right) |\nabla_\Sigma u|^2 \phi^2 \leq C_{\text{Cac}} \left(\int_\Sigma (u - k)_+^2 |\nabla_\Sigma \phi|^2 + K_{\text{Cac}}^2 \int_{\{u > k\}} \phi^2 \right),$$

where as usual all integrals should more precisely be understood on N (see 2.5). Finally, let $K_* := \max\{K_{\text{Sob}}, K_{\text{Cac}}\}$.

Lemma 2.18 (De Giorgi iteration). *Let Σ , u be as above and let $\sigma \in (0, 1)$ be fixed. There exists $\varepsilon_{\text{DG}} = \varepsilon_{\text{DG}}(n, k_*, C_{\text{Sob}}, C_{\text{Cac}}, C_c) > 0$ with the following property. Let $d \in (0, k_*]$. If $K_* R \leq 1$ and*

$$(2.6) \quad R^{-n} \int_{U_R} u^2 \leq \varepsilon_{\text{DG}}^2 (1 - \sigma)^n d^{n+2},$$

then

$$(2.7) \quad \sup_{U_{\sigma R}} u \leq d.$$

Proof. Define

$$k_\ell = d \left(1 - \frac{1}{2^\ell} \right), \quad r_\ell = \sigma R + 2^{-\ell} (1 - \sigma) R$$

for $\ell \geq 0$. Hence $k_0 = 0$ and $k_\ell \uparrow d \leq k_*$, $r_0 = R$ and $r_\ell \downarrow \sigma R$.

Since $k_\ell \leq k_*$, we can apply (2.5). Let $U_\ell := U_{r_\ell}$, $u_\ell := (u - k_\ell)_+$, and $D_\ell := U_\ell \cap \{u > k_\ell\}$. Then let

$$G_\ell := \int_{\Sigma \cap U_\ell} u_\ell^2.$$

Notice that, by definition (since both the cylinders and the truncations decrease),

$$G_{\ell+1} \leq G_\ell.$$

Now let $\phi_\ell := \phi_{r_{\ell+1}, r_\ell}$, so that

$$0 \leq \phi_\ell \leq 1, \quad \phi_\ell \equiv 1 \text{ on } U_{\ell+1}, \quad \text{spt } \phi_\ell \subset U_\ell, \quad |\nabla_\Sigma \phi_\ell| \leq \frac{C_c}{(1 - \sigma)R} 2^{\ell+1},$$

where C_c is the fixed constant from the setup.

On $D_{\ell+1}$, by definition, we have $u > k_{\ell+1}$, so that

$$1 - \frac{k_\ell}{u} \geq \frac{k_{\ell+1} - k_\ell}{k_{\ell+1}} \geq 2^{-\ell-1} \quad \text{on } D_{\ell+1}.$$

Therefore,

$$2^{-\ell-1} \int_\Sigma \phi_\ell^2 |\nabla_\Sigma u_{\ell+1}|^2 = 2^{-\ell-1} \int_{U_\ell \cap \{u > k_{\ell+1}\}} \phi_\ell^2 |\nabla_\Sigma u|^2 \leq \int_{D_\ell} \left(1 - \frac{k_\ell}{u} \right) |\nabla_\Sigma u|^2 \phi_\ell^2,$$

and, by (2.5), since $|\nabla_\Sigma \phi_\ell|^2 \leq C_c^2 4^\ell ((1 - \sigma)R)^{-2}$ and $\text{spt } \phi_\ell \subset U_\ell$, we obtain

$$\begin{aligned} \int_\Sigma \phi_\ell^2 |\nabla_\Sigma u_{\ell+1}|^2 &\leq 2^{\ell+1} C_{\text{Cac}} \left(C_c^2 4^{\ell+1} ((1 - \sigma)R)^{-2} G_\ell + K_{\text{Cac}}^2 \mathcal{H}^n(D_\ell) \right) \\ &\leq C_0 8^\ell \left(((1 - \sigma)R)^{-2} G_\ell + K_{\text{Cac}}^2 \mathcal{H}^n(D_\ell) \right), \end{aligned}$$

where, throughout this proof, C_0 denotes a constant which is allowed to change from line to line depending on $n, k_*, C_{\text{Cac}}, C_{\text{Sob}}, C_c$ only.

Since $|\nabla_\Sigma(\phi_\ell u_{\ell+1})|^2 \leq 2\phi_\ell^2 |\nabla_\Sigma u_{\ell+1}|^2 + 2u_{\ell+1}^2 |\nabla_\Sigma \phi_\ell|^2$, then

$$(2.8) \quad \int_\Sigma |\nabla(\phi_\ell u_{\ell+1})|^2 \leq C_0 8^\ell \left(((1 - \sigma)R)^{-2} G_\ell + K_{\text{Cac}}^2 \mathcal{H}^n(D_\ell) \right).$$

Notice that on D_ℓ we have $u_{\ell-1} = u - k_{\ell-1} > k_\ell - k_{\ell-1} \geq d 2^{-\ell}$, so that by Markov's inequality

$$(2.9) \quad \mathcal{H}^n(D_\ell) \leq \left(\frac{2^\ell}{d} \right)^2 \int_{D_\ell} u_{\ell-1}^2 \leq \left(\frac{2^\ell}{d} \right)^2 \int_{U_{\ell-1}} u_{\ell-1}^2 \leq 4^\ell d^{-2} G_{\ell-1},$$

as $D_\ell \subset U_\ell \subset U_{\ell-1}$. Since $u_{\ell+1} \leq u_\ell$, $G_\ell \leq G_{\ell-1}$ and $d \leq 1$, by (2.8) and (2.9), we obtain

$$(2.10) \quad \int_{\Sigma} |\nabla_{\Sigma}(u_{\ell+1}\phi_{\ell})|^2 + K_{\text{Sob}}^2 \int_{\Sigma} u_{\ell+1}^2 \phi_{\ell}^2 \leq C_0 8^{\ell} \left(((1-\sigma)R)^{-2} G_{\ell} + 4^{\ell} d^{-2} K_{*}^2 G_{\ell-1} \right) + K_{*}^2 G_{\ell} \\ \leq C_0 32^{\ell} ((1-\sigma)R)^{-2} d^{-2} G_{\ell-1},$$

where in the last inequality we used the condition $K_{*}R \leq 1$.

Case $n \geq 3$: Assume $n \geq 3$, and apply the Sobolev inequality in Lemma 2.17 to obtain

$$\left(\int_{\Sigma} |u_{\ell+1}\phi_{\ell}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_{\text{Sob}} \left(\int_{\Sigma} |\nabla_{\Sigma}(u_{\ell+1}\phi_{\ell})|^2 + K_{\text{Sob}}^2 \int_{\Sigma} u_{\ell+1}^2 \phi_{\ell}^2 \right) \\ \leq C_{\text{Sob}} \left(\int_{\Sigma} |\nabla_{\Sigma}(u_{\ell+1}\phi_{\ell})|^2 + K_{\text{Sob}}^2 G_{\ell} \right).$$

Since $u_{\ell+1} \leq u_{\ell}$ and $\text{spt } \phi_{\ell} \subset U_{\ell}$ and $\phi_{\ell} \equiv 1$ on $U_{\ell+1}$, then, by (2.10)

$$(2.11) \quad \left(\int_{U_{\ell+1}} u_{\ell+1}^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \leq C_0 32^{\ell} ((1-\sigma)R)^{-2} d^{-2} G_{\ell-1}.$$

By Hölder's inequality and (2.9)

$$(2.12) \quad G_{\ell+1} = \int_{D_{\ell+1}} u_{\ell+1}^2 \leq \left(\int_{U_{\ell+1}} u_{\ell+1}^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \mathcal{H}^n(D_{\ell+1})^{\frac{2}{n}} \\ \leq C_0 32^{\ell} ((1-\sigma)R)^{-2} d^{-2} G_{\ell-1} (4^{\ell+1} d^{-2} G_{\ell})^{\frac{2}{n}} \\ \leq C_0 128^{\ell} ((1-\sigma)R)^{-2} d^{-2-\frac{4}{n}} G_{\ell-1}^{1+\frac{2}{n}}.$$

Case $n = 2$: For $n = 2$, the Sobolev inequality in Lemma 2.17 gives

$$\left(\int_{\Sigma} u_{\ell+1}^4 \phi_{\ell}^4 \right)^{1/2} \leq C_{\text{Sob}} \left(\int_{\Sigma} u_{\ell+1}^2 \phi_{\ell}^2 \right)^{1/2} \left(\int_{\Sigma} |\nabla_{\Sigma}(u_{\ell+1}\phi_{\ell})|^2 + K_{\text{Sob}}^2 \int_{\Sigma} u_{\ell+1}^2 \phi_{\ell}^2 \right)^{1/2}$$

so that

$$\left(\int_{U_{\ell+1}} u_{\ell+1}^4 \right)^{1/2} \leq C_{\text{Sob}} G_{\ell}^{1/2} \left(C_0 32^{\ell} ((1-\sigma)R)^{-2} d^{-2} G_{\ell-1} \right)^{1/2} \\ \leq C_0 32^{\ell} ((1-\sigma)R)^{-1} d^{-1} G_{\ell-1}.$$

By Hölder's inequality,

$$(2.13) \quad G_{\ell+1} \leq \left(\int_{U_{\ell+1}} u_{\ell+1}^4 \right)^{1/2} \mathcal{H}^n(D_{\ell+1})^{1/2} \\ \leq \left(C_0 32^{\ell} ((1-\sigma)R)^{-1} d^{-1} G_{\ell-1} \right) \left(4^{\ell+1} d^{-2} G_{\ell} \right)^{1/2} \\ \leq C_0 128^{\ell} ((1-\sigma)R)^{-1} d^{-2} G_{\ell-1}^{3/2}.$$

Notice that (2.12) and (2.13) are of the same form. More concretely, if we set

$$p_n := \begin{cases} \frac{2}{n} & \text{if } n \geq 3, \\ \frac{1}{2} & \text{if } n = 2, \end{cases}$$

and $S_j := ((1 - \sigma)R)^{-n} d^{-n-2} G_{2j}$, then by letting $\ell = 2j + 1$ in (2.12) and (2.13) respectively, we obtain

$$S_{j+1} \leq C_0 128^{2j+1} S_j^{1+p_n}$$

for $j \geq 0$. The standard nonlinear iteration lemma (see [Bel25, Lemma B.1]) implies that there exists a constant $\varepsilon_{\text{DG}} = \varepsilon_{\text{DG}}(n, k_*, C_{\text{Sob}}, C_{\text{Cac}}, C_c) > 0$ such that $S_j \rightarrow 0$ as $j \rightarrow \infty$ provided $S_0 \leq \varepsilon_{\text{DG}}^2$. Since $\{G_\ell\}_\ell$ is a non-increasing sequence, this means that

$$G_\ell \rightarrow 0 \text{ as } \ell \rightarrow \infty, \text{ provided that } G_0 \leq \varepsilon_{\text{DG}}^2 ((1 - \sigma)R)^n d^{n+2},$$

as required. $\square$

We shall always apply Lemma 2.18 in the following form.

Corollary 2.19. *Under the assumptions of Lemma 2.18, let*

$$E(R) := \left(R^{-n} \int_{U_R} u^2 \right)^{\frac{1}{2}}.$$

Then there exist $\varepsilon_ = \varepsilon_*(n, k_*, C_{\text{Sob}}, C_{\text{Cac}}, C_c) > 0$ and $C_* = C_*(n, k_*, C_{\text{Sob}}, C_{\text{Cac}}, C_c) > 0$ such that if*

$$K_* R \leq 1 \quad \text{and} \quad E(R) \leq \varepsilon_*(1 - \sigma)^{n/2},$$

then $\sup_{U_{\sigma R}} u \leq C_(1 - \sigma)^{-\frac{n}{n+2}} E(R)^{\frac{2}{n+2}}$.*

Proof. This follows from Lemma 2.18. Indeed, let $\varepsilon_* = \varepsilon_{\text{DG}} k_*^{\frac{n+2}{2}}$ and $C_* := \varepsilon_{\text{DG}}^{-\frac{2}{n+2}}$. Then, if $E(R) \leq \varepsilon_*(1 - \sigma)^{n/2}$, take $d := \varepsilon_{\text{DG}}^{-\frac{2}{n+2}} (1 - \sigma)^{-\frac{n}{n+2}} E(R)^{\frac{2}{n+2}}$ and apply Lemma 2.18. $\square$

3. BOUNDARY-PARALLEL SHEETING

3.1. Geometric setup. Throughout this section, let us fix $p \in \partial M$, and let $\rho_0 > 0$ be such that

- the intrinsic geodesic balls $P_R = B_R^{\partial M}(p)$ are smooth and relatively compact in ∂M for $0 \leq R \leq 2\rho_0$;
- the normal exponential map $F_{\partial M}$ of ∂M is a diffeomorphism of $P_{2\rho_0} \times [0, 2\rho_0)$ onto a relatively open set.

For $0 \leq \delta \leq 2\rho_0$, $0 < R \leq 2\rho_0$, define $\mathcal{N}_\delta^+(P_R) := F_{\partial M}(P_R \times [0, \delta))$. In particular, the distance function $d_{\partial M}$ to ∂M is smooth on $\mathcal{N}_{2\rho_0}^+(P_{2\rho_0})$.

Let $\Lambda \geq 0$ be such that

$$(3.1) \quad |\nabla_g^2 d_{\partial M}|^2 + |\nabla_g^3 d_{\partial M}| + |\text{Rm}_g| \leq \Lambda^2 \quad \text{in } \mathcal{N}_{2\rho_0}^+(P_{2\rho_0}).$$

Now let us assume as in Theorem 1.11 that Σ is a two-sided free-boundary minimal hypersurface in $\mathcal{N}_{\rho_0}^+(P_R)$ such that $\bar{\Sigma} \subset \mathcal{N}_\delta^+(P_R)$ for some $\delta \in (0, \rho_0)$, $R \in (0, R_0)$, where $R_0 \in (0, \rho_0)$ is to be specified later depending on n and Λ only.

Definition 3.1 (Tilt function). Let $X := \nabla_\Sigma d_{\partial M}$, $g = |X|_g$, and $\theta := g(\nu, \nabla^g d_{\partial M})$. We shall refer to g as the *tilt function* on Σ with respect to ∂M .

We start by computing the partial differential inequality satisfied by the tilt on Σ .

Lemma 3.2 (Differential inequalities for the tilt). *Let (M, g) and P_R be as above. Additionally, let Σ be a smooth two-sided free-boundary minimal hypersurface in M as in Section 2 and assume that $\Sigma \subset \mathcal{N}_\delta^+(P_R)$. Let ν be a choice of unit normal to Σ , A its second fundamental form, and let X, g, θ be as in Definition 3.1. Then $g^2 + \theta^2 = 1$. Set $B := \nabla_g^2 d_{\partial M}|_{T\Sigma}$. Then the following hold.*

First, on $\{g > 0\}$, if $v := X/g$, then

$$\nabla_{\Sigma} g = \theta A(\cdot, v) + B(\cdot, v)$$

and, for every $\gamma > 0$,

$$|\nabla_{\Sigma} g|^2 \leq (1 + \gamma)(1 - g^2)|A(\cdot, v)|^2 + C(\gamma)|B|^2.$$

Second, assuming that (3.1) holds, then for every $\gamma > 0$ on $\{g > 0\}$

$$g\Delta_{\Sigma} g \geq (1 - \gamma)|A|^2(1 - g^2) - \left(\frac{n-1}{n} + \gamma\right)|A|^2 - \gamma|A|^2 g^2 - C(n, \gamma)\Lambda^2.$$

Proof. Since $d_{\partial M}$ is the signed g-distance to P in $\mathcal{N}_{\rho_0}^+(P)$, we have $|\nabla^g d_{\partial M}|_g = 1$ in the tubular neighbourhood. Thus $\nabla^g d_{\partial M} = X + \theta\nu$ along Σ , and so $g^2 + \theta^2 = 1$.

The identity $\nabla_{\Sigma}^2 d_{\partial M} = B + \theta A$ follows from the relation between ambient and intrinsic Hessians. Indeed, for $Y, Z \in T\Sigma$,

$$\nabla_{\Sigma}^2 d_{\partial M}(Y, Z) = \nabla_g^2 d_{\partial M}(Y, Z) + g(\nabla^g d_{\partial M}, \nu)A(Y, Z) = B(Y, Z) + \theta A(Y, Z).$$

By taking $Z = X$ in this identity, we immediately obtain that

$$gYg = \frac{1}{2}Y(g^2) = \nabla_{\Sigma}^2 d_{\partial M}(Y, X) = B(Y, X) + \theta A(Y, X).$$

On $\{g > 0\}$, writing $X = gv$, this becomes $Yg = \theta A(Y, v) + B(Y, v)$ as required.

Moreover, taking norms and using $\theta^2 = 1 - g^2$, we obtain

$$|\nabla_{\Sigma}^2 d_{\partial M}|^2 = |A|^2(1 - g^2) + 2\theta\langle A, B \rangle + |B|^2.$$

The classical Bochner formula on Σ gives

$$\Delta_{\Sigma} \frac{g^2}{2} = |\nabla_{\Sigma}^2 d_{\partial M}|^2 + \langle \nabla_{\Sigma} \Delta_{\Sigma} d_{\partial M}, X \rangle + \text{Ric}_{\Sigma}(X, X).$$

Since $\Delta_{\Sigma}(g^2/2) = g\Delta_{\Sigma} g + |\nabla_{\Sigma} g|^2$, we get

$$g\Delta_{\Sigma} g = |\nabla_{\Sigma}^2 d_{\partial M}|^2 + \langle \nabla_{\Sigma} \Delta_{\Sigma} d_{\partial M}, X \rangle + \text{Ric}_{\Sigma}(X, X) - |\nabla_{\Sigma} g|^2.$$

The contracted Gauss equation gives, since $H = 0$,

$$\text{Ric}_{\Sigma}(X, X) = \text{Ric}_g(X, X) - \text{Rm}_g(X, \nu, X, \nu) - |A(\cdot, X)|^2.$$

Combining this with the expansion of $|\nabla_{\Sigma}^2 d_{\partial M}|^2$ yields the exact Bochner identity.

Now let $\gamma > 0$. From the identity $\nabla_{\Sigma} g = \theta A(\cdot, v) + B(\cdot, v)$ and Young's inequality, we obtain

$$|\nabla_{\Sigma} g|^2 \leq (1 + \gamma)(1 - g^2)|A(\cdot, v)|^2 + C(\gamma)|B|^2.$$

By definition, $|A(\cdot, X)|^2 = g^2|A(\cdot, v)|^2$, so that

$$|\nabla_{\Sigma} g|^2 + |A(\cdot, X)|^2 \leq (1 + \gamma(1 - g^2))|A(\cdot, v)|^2 + C(\gamma)|B|^2.$$

Since $H = 0$, A is trace-free. Hence, for every unit vector v , we have $|A(\cdot, v)|^2 \leq \frac{n-1}{n}|A|^2$. Therefore,

$$|\nabla_{\Sigma} g|^2 + |A(\cdot, X)|^2 \leq \left(\frac{n-1}{n} + \gamma\right)|A|^2 + C(\gamma)|B|^2.$$

Again using Young's inequality, we also bound

$$2|\theta\langle A, B \rangle| \leq \gamma|A|^2\theta^2 + C(\gamma)|B|^2 = \gamma|A|^2(1 - g^2) + C(\gamma)|B|^2,$$

so that, after substituting into the exact Bochner identity, we obtain

$$\begin{aligned} g\Delta_\Sigma g &\geq (1-\gamma)|A|^2(1-g^2) - \left(\frac{n-1}{n} + \gamma\right)|A|^2 - C(\gamma)|B|^2 \\ &\quad + \langle \nabla_\Sigma \Delta_\Sigma d_{\partial M}, X \rangle + \text{Ric}_g(X, X) - \text{Rm}_g(X, \nu, X, \nu). \end{aligned}$$

Finally, since Σ is minimal, $\Delta_\Sigma d_{\partial M} = \text{tr}_\Sigma \nabla_g^2 d_{\partial M} = \Delta_g d_{\partial M} - \nabla_g^2 d_{\partial M}(\nu, \nu)$. Differentiating this expression tangentially gives terms involving $\nabla_g^2 d_{\partial M}$, $\nabla_g^3 d_{\partial M}$, and one derivative of ν , hence one factor of $|A|$, namely

$$|\langle \nabla_\Sigma \Delta_\Sigma d_{\partial M}, X \rangle| \leq (C(n)|\nabla_g^3 d_{\partial M}| + 2|A||\nabla_g^2 d_{\partial M}|)g.$$

By Young's inequality, $2|A||\nabla_g^2 d_{\partial M}|g \leq \gamma|A|^2 g^2 + C(\gamma)|\nabla_g^2 d_{\partial M}|^2$. The curvature terms can be estimated by

$$|\text{Ric}_g(X, X) - \text{Rm}_g(X, \nu, X, \nu)| \leq C(n)|\text{Rm}_g|g^2.$$

Assuming (3.1) holds, this gives the final estimate, since $0 \leq g \leq 1$ by definition. $\square$

Next we compute the normal derivative of the tilt function along ∂M .

Lemma 3.3. *Let $\Sigma, M, d_{\partial M}$, and g be as in Lemma 3.2. Let η be the inward unit conormal to $\partial\Sigma$ in Σ . Then*

$$g = 1 \quad \text{and} \quad \partial_\eta g = 0 \quad \text{on } \partial\Sigma.$$

Proof. By definition of free-boundary minimal hypersurface, η coincides with the normal $\mathbf{n}$ to ∂M along $\partial\Sigma$. Since P_R is a relatively open subset of ∂M and $\partial\Sigma \subset P_R$, then $\eta = \nabla^g d_{\partial M}|_{\partial\Sigma}$.

Hence, $\nabla^g d_{\partial M}$ is tangential to Σ along $\partial\Sigma$ and so, by definition, $\nabla^g d_{\partial M}|_{\partial\Sigma} = \nabla_\Sigma d_{\partial M}|_{\partial\Sigma}$. In particular, then, for every $x \in \partial\Sigma$

$$g(x) = |\nabla_\Sigma d_{\partial M}(x)| = |\nabla^g d_{\partial M}(x)| = 1,$$

i.e. $g|_{\partial\Sigma} \equiv 1$. Since $g^2 + \theta^2 = 1$ everywhere on Σ , then $\theta|_{\partial\Sigma} \equiv 0$. Thus, on $\partial\Sigma$

$$\partial_\eta g = g\partial_\eta g = \frac{1}{2}\partial_\eta g^2 = -\frac{1}{2}\partial_\eta \theta^2 = -\theta\partial_\eta \theta = 0$$

as θ vanishes on $\partial\Sigma$. $\square$

3.2. Caccioppoli-type inequality. We shall now prove the intrinsic (weak) Caccioppoli inequality that will be used later to implement the iteration scheme à la De Giorgi.

Lemma 3.4 (Caccioppoli inequality). *Let Σ be a stable singular free-boundary minimal hypersurface in M and assume that $\bar{\Sigma} \subset \mathcal{N}_\delta^+(P_R)$. Assume that (3.1) holds. Then there exists a positive dimensional constant $C(n) > 0$ such that for every $k \in [0, 1/(4n)]$ and every $\varphi \in C_c^{0,1}(\mathcal{N}_{\rho_0}^+(P_R))$, we have*

$$(3.2) \quad \frac{1}{2n} \int_{\{g>k\}} |A|^2 \left(1 - \frac{k}{g}\right) \varphi^2 \leq C(n) \left(\int_\Sigma (g-k)_+^2 |\nabla_\Sigma \varphi|^2 + \Lambda^2 \int_{\{g>k\}} \varphi^2 \right).$$

Consequently,

$$(3.3) \quad \int_{\{g>k\}} |\nabla_\Sigma g|^2 \left(1 - \frac{k}{g}\right) \varphi^2 \leq C(n) \left(\int_\Sigma (g-k)_+^2 |\nabla_\Sigma \varphi|^2 + \Lambda^2 \int_{\{g>k\}} \varphi^2 \right).$$

Proof. We shall first prove (3.2) and (3.3) for test functions $\varphi \in C_c^{0,1}(\Sigma)$, and then extend to compactly supported ambient Lipschitz functions by an excision argument, similar to the one in Lemma 2.14.

Let $\Omega_k := \{g > k\}$. By Lemma 2.16 for every compactly supported Lipschitz function ζ on Σ and a fixed $\alpha > 0$ to be determined later, we have

$$\int_{\Sigma} |A|^2 \zeta^2 \leq (1 + \alpha) \int_{\Sigma} |\nabla_{\Sigma} \zeta|^2 + C(n, \alpha) \Lambda^2 \int_{\Sigma} \zeta^2$$

where $C(n, \alpha)$ only depends on n, α . Let now $\zeta = (g - k)_+ \varphi$ to get

$$\int_{\Sigma} |A|^2 (g - k)_+^2 \varphi^2 \leq (1 + \alpha) \int_{\Sigma} |\nabla_{\Sigma} ((g - k)_+ \varphi)|^2 + C(n, \alpha) \Lambda^2 \int_{\Sigma} (g - k)_+^2 \varphi^2.$$

After integration by parts, we have

$$\int_{\Sigma} |\nabla_{\Sigma} ((g - k)_+ \varphi)|^2 = - \int_{\Omega_k} (g - k) \Delta_{\Sigma} g \varphi^2 + \int_{\Sigma} (g - k)_+^2 |\nabla_{\Sigma} \varphi|^2,$$

and the boundary term vanishes by Lemma 3.3. Hence,

$$\begin{aligned} \int_{\Omega_k} |A|^2 (g - k)^2 \varphi^2 &\leq -(1 + \alpha) \int_{\Omega_k} (g - k) \Delta_{\Sigma} g \varphi^2 + (1 + \alpha) \int_{\Sigma} (g - k)_+^2 |\nabla_{\Sigma} \varphi|^2 \\ &\quad + C(n, \alpha) \Lambda^2 \int_{\Sigma} (g - k)_+^2 \varphi^2. \end{aligned}$$

Notice that on Ω_k , one has

$$(g - k) \Delta_{\Sigma} g = \left(1 - \frac{k}{g}\right) g \Delta_{\Sigma} g,$$

so we can use the differential inequality for the tilt in Lemma 3.2 to obtain for $\gamma > 0$ to be determined later

$$-(g - k) \Delta_{\Sigma} g \leq - \left(1 - \frac{k}{g}\right) \left[\left((1 - \gamma)(1 - g^2) - \frac{n - 1}{n} - \gamma - \gamma g^2 \right) |A|^2 - C(n, \gamma) \Lambda^2 \right].$$

Thus, after moving all the terms containing $|A|^2$ to the left hand side and using the obvious inequality $0 \leq (g - k)_+ \leq 1$, we get

$$\int_{\Omega_k} \mathcal{Q}_{\alpha, \gamma} |A|^2 \varphi^2 \leq (1 + \alpha) \int_{\Sigma} (g - k)_+^2 |\nabla_{\Sigma} \varphi|^2 + (C(n, \alpha) + C(n, \gamma)) \Lambda^2 \int_{\Omega_k} \varphi^2,$$

where

$$\mathcal{Q}_{\alpha, \gamma} := (g - k)^2 + (1 + \alpha) \left(1 - \frac{k}{g}\right) \left((1 - \gamma)(1 - g^2) - \frac{n - 1}{n} - \gamma - \gamma g^2 \right).$$

By direct computation,

$$\mathcal{Q}_{\alpha, \gamma} = \left(1 - \frac{k}{g}\right) \left(\frac{1 + \alpha}{n} - kg - \alpha g^2 - 2(1 + \alpha)\gamma \right).$$

We can choose $\alpha = \alpha_n := 1/(8n)$ and $\gamma = \gamma_n := 1/(16n(1 + \alpha_n))$ to obtain

$$\frac{1 + \alpha}{n} - kg - \alpha g^2 - 2(1 + \alpha)\gamma \geq \frac{1}{2n}$$

since $k \in [0, 1/(4n)]$ and $0 \leq g \leq 1$ by definition. Inequality (3.2) then follows.

As for (3.3), recall by Lemma 3.2 on the open set $\{g > 0\}$ we have $\nabla_{\Sigma} g = \theta A(v, \cdot) + B(v, \cdot)$ for the unit vector $v = g^{-1}X$. Therefore, on $\Sigma \subset \mathcal{N}_{\rho_0}^+(P_R)$

$$|\nabla_{\Sigma} g|^2 \leq 2\theta^2 |A|^2 + 2|B|^2 \leq 2|A|^2 + 2|\nabla_g^2 d_{\partial M}|^2 \leq 2|A|^2 + 2\Lambda^2,$$

and (3.3) then follows from (3.2).

Finally, apply Corollary B.6 on a compact neighbourhood of $\text{spt } \varphi$: there exist Lipschitz cutoff functions χ_j , each supported away from $\mathcal{S}$, such that $0 \leq \chi_j \leq 1$, $\chi_j \rightarrow 1$ locally on Σ , and $\int_{\Sigma} |\nabla_{\Sigma} \chi_j|^2 \rightarrow 0$. We can then apply the smooth inequality (3.2) with test function $\chi_j \varphi$ to get

$$\frac{1}{2n} \int_{\{g > k\}} |A|^2 \left(1 - \frac{k}{g}\right) \chi_j^2 \varphi^2 \leq C(n) \int_{\Sigma} (g - k)_+^2 |\nabla_{\Sigma}(\chi_j \varphi)|^2 + C(n) \Lambda^2 \int_{\{g > k\}} \chi_j^2 \varphi^2.$$

Since

$$|\nabla_{\Sigma}(\chi_j \varphi)|^2 \leq 2\chi_j^2 |\nabla_{\Sigma} \varphi|^2 + 2\varphi^2 |\nabla_{\Sigma} \chi_j|^2,$$

and $0 \leq (g - k)_+ \leq 1$, the contribution of the second term is bounded by $\|\varphi\|_{L^\infty}^2 \int_{\Sigma} |\nabla_{\Sigma} \chi_j|^2$, which tends to zero. All other terms pass to the limit by Fatou's lemma, the dominated convergence theorem, and the local area bound, cf. Remark 2.7. Thus, letting $j \rightarrow \infty$, we obtain the asserted estimate with $|A|^2$ on the left-hand side. Then (3.3) follows exactly as above. $\square$

3.3. Proof of Theorem 1.11. We are now able to prove Theorem 1.11 by applying the iteration lemma in Section 2.3.

Let us assume that (3.1) holds. Define $K_0(n, \Lambda) := \max\{K_{\text{Sob}}(n, \Lambda), K_{\text{Cac}}(n, \Lambda)\}$, and set

$$(3.4) \quad \delta_0 := \frac{1}{2} \rho_0, \quad R_0 := \frac{1}{2} \min\{\rho_0, K_0^{-1}\}.$$

Let Σ be a stable singular free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset \mathcal{N}_\delta^+(P_R)$ for some $\delta \in (0, \delta_0)$, $R \in (0, R_0)$.

Then, for $0 < r \leq R$ define the *Fermi cylinder* C_r by

$$(3.5) \quad C_r := F_{\partial M}(P_r \times [0, 2\delta)) = \mathcal{N}_{2\delta}^+(P_r),$$

and consider the following *tilt excess* quantity:

$$(3.6) \quad E_{\Sigma, P}(r) := \left(r^{-n} \int_{\Sigma \cap C_r} g^2 \right)^{1/2}$$

where g is the *tilt function* in Definition 3.1.

Let us apply Lemma 2.18 with $u = g$ and $k_* = 1/(4n)$. Notice that Lemma 3.4 provides the necessary Caccioppoli-type inequality (2.5). Therefore, by Corollary 2.19, there exist $\varepsilon_0 = \varepsilon_0(n, \Lambda) > 0$ and $C_0 = C_0(n, \Lambda) > 0$ such that

$$\sup_{\Sigma \cap C_{\sigma R}} g \leq C_0 (1 - \sigma)^{-\frac{n}{n+2}} E_{\Sigma, P}(R)^{\frac{2}{n+2}}$$

provided $E_{\Sigma, P}(R)^2 \leq \varepsilon_0^2 (1 - \sigma)^n$, which is the desired statement.

3.4. Proof of Theorem 1.12. We shall now prove our first sheeting theorem, Theorem 1.12. The proof is similar to [Bel25, Section 5], to which we shall often refer.

Assume $\bar{\Sigma} \subset \mathcal{N}_\delta^+(P_R) = F_{\partial M}(P_R \times [0, \delta))$, and recall that $F_{\partial M}$ is a diffeomorphism $P_R \times [0, \delta) \rightarrow \mathcal{N}_\delta^+(P_R)$. Let the inverse map $F_{\partial M}^{-1} : \mathcal{N}_\delta^+(P_R) \rightarrow P_R \times [0, \delta)$ be given by $(\pi_{\partial M}, d_{\partial M})|_{\mathcal{N}_\delta^+(P_R)}$, where $\pi_{\partial M}$ is the nearest point projection map onto ∂M .

By Theorem 1.11 with $\sigma = 3/4$, we have that

$$(3.7) \quad \sup_{\Sigma \cap C_{3R/4}} g \leq C_0(n, \Lambda) E_{\Sigma, P}(R)^{\frac{2}{n+2}}.$$

By decreasing ε_0 if necessary, we may assume $\sup_{\Sigma \cap C_{3R/4}} g \leq \frac{1}{4}$.

Step 0: $\partial\Sigma \cap P_{3R/4} = \emptyset$. Suppose $x \in \partial\Sigma \cap P_{3R/4}$. By definition, the inner conormal $\eta(x)$ of $\partial\Sigma$ at x equals $\mathbf{n}(x) = \nu_P(x)$. Hence, $\nu_P(x) \in T_x\Sigma$ and $g(x) = |\text{proj}_g^{T_x\Sigma} \nabla^g d_{\partial M}(x)| = |\text{proj}_g^{T_x\Sigma} \nu_P(x)| = 1$, contradicting (3.7).

Step 1: Local graphicality. Let $\iota : N \rightarrow M \setminus \mathcal{S}$ parametrise Σ . At every point of $\mathcal{N}_\delta^+(P_R)$,

$$(3.8) \quad \text{Ker } D\pi_{\partial M} = \text{span}\{\nabla^g d_{\partial M}\}.$$

By (3.7) and (3.8), the differential of $\pi_{\partial M} \circ \iota$ is invertible at every $z \in N$ such that $\iota(z) \in C_{3R/4}$. Thus the argument in [Bel25, Proof of Theorem 6] shows that there is a neighbourhood of z in N the image of which is a smooth Fermi graph over an open subset of $P_{3R/4}$.

Step 2: Finite covering away from the projected singular set. Let

$$\Gamma := \pi_{\partial M}(\mathcal{S} \cap \overline{\mathcal{N}_\delta^+}(P_{3R/4})) \subset \bar{P}_{3R/4}, \quad B := P_{3R/4} \setminus \Gamma.$$

Since $\mathcal{H}^{n-2}(\mathcal{S} \cap \overline{\mathcal{N}_\delta^+}(P_{3R/4})) < \infty$ and $\pi_{\partial M}$ is a Lipschitz map in $\mathcal{N}_{\rho_0}^+(P_R)$, then $\mathcal{H}^{n-1}(\Gamma) = 0$. Since $\mathcal{S}$ is closed, it is easy to see that Γ is relatively closed in $P_{3R/4}$. Then B is open in $P_{3R/4}$ and path connected (see [Bel25, Proof of Theorem 5]). Define

$$N_0 := \{z \in N : \iota(z) \in C_{3R/4}, \pi_{\partial M}(\iota(z)) \in B\}$$

and $\hat{\pi} := \pi_P \circ \iota|_{N_0} : N_0 \rightarrow B$. By *Step 1*, $\hat{\pi}$ is a local diffeomorphism. Because of properness of ι as an immersion into $M \setminus \mathcal{S}$, one can check that $\hat{\pi}$ is also proper. Since B is connected, each connected component of N_0 is therefore mapped onto the whole of B . When restricted to each connected component of N_0 , $\hat{\pi}$ is a proper local diffeomorphism between connected smooth manifolds, so it is a smooth covering map by [Lee13, Proposition 4.46]. In particular, for each $y \in B$, $\hat{\pi}^{-1}(y)$ is finite. Its cardinality is a locally constant function on B , so it is constant as B is connected. Let this cardinality be Q .

Step 3: Lipschitz estimate and extension across Γ . By *Step 2*, for each $y \in B$ the normal fibre $F(\{y\} \times [0, \delta))$ has exactly Q preimages in N_0 . Order the heights of their images under ι counting repetitions, and write $0 \leq u_1(y) \leq u_2(y) \leq \dots \leq u_Q(y)$, for $y \in B$, where the functions u_j are locally Lipschitz. Then,

$$(3.9) \quad \Sigma \cap \pi_{\partial M}^{-1}(B) = \cup_{j=1}^Q F_{\partial M}(\text{graph}(u_j)).$$

Write $F_{\partial M}^* g = dt^2 + h_t$, where h_t is a smooth family of metrics on P_R with $h_0 = g|_{P_R}$. It is easy to check that for a smooth graph $G[u] = F_{\partial M}(\{(y, u(y)) : y \in P_R\})$ on P_R with $|\nabla_{G[u]} d_{\partial M}| \leq c \in (0, 1)$, then $|\nabla_{\partial M} u| \leq C_{\text{Fermi}}(n, \Lambda) c(1 - c^2)^{-1/2}$, where C_{Fermi} controls the uniform equivalence of the metrics $(h_t)_{t \in [0, \delta]}$ and depends on n and Λ by (3.1).

Since we can approximate curves in $P_{3R/4}$ by curves avoiding Γ , we conclude that

$$\text{Lip}_{\partial M}(u_j; B) \leq 2C_{\text{Fermi}} \sup_{\Sigma \cap C_{3R/4}} g \leq C(n, \Lambda) E_{\Sigma, P}(R)^{\frac{2}{n+2}}.$$

By density of B in $P_{3R/4}$, every u_j extends uniquely to a Lipschitz function (still denoted u_j) on the whole of $P_{3R/4}$, with the same bound on the Lipschitz constant, i.e.

$$(3.10) \quad \text{Lip}_{\partial M}(u_j) \leq C(n, \Lambda) E_{\Sigma, P}(R)^{\frac{2}{n+2}}.$$

By density of the regular set in $\bar{\Sigma}$ and of B in $P_{3R/4}$, we obtain

$$\bar{\Sigma} \cap \mathcal{N}_\delta^+(P_{3R/4}) = \cup_{j=1}^Q F_{\partial M}(\text{graph}(u_j))$$

for the extended functions u_j .

We can now prove (i) and (ii).

- (i) Assume $\mathcal{H}^{n-2}(\mathcal{S}) = 0$. Since $\pi_{\partial M}$ is Lipschitz on $\mathcal{N}_{\rho_0}^+(P_R)$, then necessarily $\mathcal{H}^{n-2}(\Gamma) = 0$. By [SW16, Appendix A], then $B = P_{3R/4} \setminus \Gamma$ is simply connected. Therefore every connected component of the covering $\hat{\pi} : N_0 \rightarrow B$ is one-sheeted. The components determine Q globally defined, not necessarily ordered Lipschitz functions $v_1, \dots, v_Q : B \rightarrow [0, \delta)$ such that

$$\Sigma \cap \pi_{\partial M}^{-1}(B) = \cup_{j=1}^Q F_{\partial M}(\text{graph}(v_j)).$$

Arguing as above, for every $j \in 1, 2, \dots, Q$, we can extend v_j to a Lipschitz function defined on the whole of $P_{3R/4}$, preserving the Lipschitz constant. Clearly, $\mathcal{H}^{n-1}(\Gamma) = 0$. Since $\partial \Sigma \cap P_{3R/4} = \emptyset$, the Lipschitz functions v_j are weak solutions of the minimal surface equation on the bounded open set $B = P_{3R/4} \setminus \Gamma$ with a uniform bound on the Lipschitz constant. Since the metrics $(h_t)_{t \in [0, \delta]}$ are uniformly equivalent, in local coordinates the area integrand $\mathcal{L}(y, z, q) = (\det h_z(y))^{1/2} (1 + h_z^{ij}(y) q_i q_j)^{1/2}$ is uniformly elliptic. Therefore, the removal of singularities result in Lemma A.2 applies, and $\mathcal{S} \cap \mathcal{N}_{\delta}^+(P_{3R/4}) = \emptyset$.

Finally, the gradient bound $\|\nabla v_j\|_{C^{1,\alpha}(P_{R/2})}$ follows from the combination of De Giorgi–Nash–Moser theory and Schauder theory.

- (ii) Assume instead Σ is embedded. Then for every $y \in B$ the points of Σ lying in the normal fibre are distinct, i.e. $0 \leq u_1 < \dots < u_Q$ on B . Therefore, each u_j is smooth on B . Since $\mathcal{H}^{n-1}(\Gamma) = 0$, as above by Lemma A.2 each u_j extends to a smooth solution of the minimal surface equation on the whole of $P_{3R/4}$. Weak ordering is preserved, and, indeed, strict ordering by the strong maximum principle for the minimal surface equation. Thus $\bar{\Sigma} \cap \mathcal{N}_{\delta}^+(P_{3R/4})$ is a union of pairwise disjoint smooth graphs. In particular $\mathcal{S} \cap \mathcal{N}_{\delta}^+(P_{3R/4}) = \emptyset$. The gradient estimates then follow from standard elliptic theory as in (i).

4. BOUNDARY-ORTHOGONAL SHEETING

4.1. Geometric setup. Let M be an $(n+1)$ -dimensional Riemannian manifold-with-boundary as in Section 2, and assume $\partial M \neq \emptyset$. Let P be a (smooth) two-sided, embedded, n -dimensional manifold-with-boundary in M that is diffeomorphic to a half-disc and whose boundary lies in ∂M . In other words, P is the image of a smooth embedding e from $\mathbb{B}_+^n(0) = \{x \in \mathbb{R}^n : |x| < 1, x_n \geq 0\}$ into M and $\partial_0 P = e(\partial_0 \mathbb{B}_+^n(0)) \subset \partial M$, where $\partial_0 \mathbb{B}_+^n(0) = \{x \in \mathbb{R}^n : |x| < 1, x_n = 0\}$. We shall further assume that $P \cap \partial M = \partial_0 P$ and P meets ∂M orthogonally along $\partial_0 P$. Let ν_P denote a smooth choice of unit normal on P . Then by orthogonality to ∂M along $\partial_0 P$, we have $g(\nu_P, \mathbf{n}) = 0$ on $\partial_0 P$.

Let $p \in \partial_0 P$, and, for $0 < R < \frac{1}{2} \text{inj}_p(P)$, let P_R denote the intrinsic half-ball $B_R^P(p)$ in P centred at p .

Definition 4.1 (Adapted ρ -neighbourhood). Let $\rho > 0$. An *adapted ρ -neighbourhood* of P_R in M consists of a relatively open neighbourhood $U \subset M$ of P_{2R} and a smooth function $z_P : U \rightarrow \mathbb{R}$ with the following properties:

- (i) $d_{\partial M}$ is smooth in U ;
- (ii) $P \cap U = \{z_P = 0\}$ and $\nabla^g z_P$ is a positive multiple of ν_P on $P \cap U$;
- (iii) $\frac{1}{2} \leq |\nabla^g z_P| \leq 2$ in U ;
- (iv) $\langle \nabla^g z_P, \mathbf{n} \rangle = 0$ in $U \cap \partial M$.
- (v) the flow Φ_t^Z of the vector field $Z = \frac{\nabla^g z_P}{|\nabla^g z_P|}$ defines a diffeomorphism F_Z of $P_{2R} \times (-2\rho, 2\rho)$ onto a relatively open (compactly contained in U) neighbourhood of P_{2R} in M by

$$(4.1) \quad F_Z(y, t) := \Phi_t^Z(y) \quad \text{for } (y, t) \in P_{2R} \times (-2\rho, 2\rho).$$

Remark 4.2. The essential datum in an adapted neighbourhood is the vector field Z . One could instead prescribe a smooth unit vector field on a relatively open neighbourhood U of P_{2R} , equal to ν_P on P_{2R} and tangent to ∂M along $\partial M \cap U$, whose flow satisfies condition (v) of Definition 4.1. With the derivative bounds (4.6), the flow estimate (4.7)–(4.8), and the ambient geometric bounds used below, the ε -regularity and sheeting arguments apply without requiring Z to be a normalized gradient.

In particular, the tilt and Caccioppoli estimates involve only the first two derivatives of Z , whereas deriving these bounds from $Z = \nabla^g z_P / |\nabla^g z_P|$ requires control of three derivatives of z_P . The function z_P therefore provides a convenient construction, rather than an essential ingredient of the argument.

We now show that for every P as above, there exists $\rho > 0$ such that the half disc P_ρ has an adapted ρ -neighbourhood.

Lemma 4.3. *Let $P \subset M$ be as above and let $p \in \partial_0 P$. Then there exists $0 < \rho < \frac{1}{2} \text{inj}_p(P)$ sufficiently small such that P_ρ has an adapted ρ -neighbourhood U in the sense of Definition 4.1.*

Proof. Consider the smooth embedded relatively compact submanifold $\Gamma := \partial_0 P$ of ∂M . By [Gra04, Section 3.1], there exists $\varrho_0 > 0$ such that the normal exponential map F_Γ of Γ in ∂M is a diffeomorphism of $\Gamma \times (-\varrho_0, \varrho_0)$ onto an open set O_Γ in ∂M . Since O_Γ has compact closure in ∂M , then, as mentioned in Section 2, after shrinking ϱ_0 if needed, the normal exponential map of ∂M is a diffeomorphism of $O_\Gamma \times [0, \varrho_0)$ into a normal collar neighbourhood U_Γ of O_Γ in M . In particular, the composition of the two normal exponential maps gives a diffeomorphism Ψ of $\Gamma \times (-\varrho_0, \varrho_0) \times [0, \varrho_0)$ onto $U_\Gamma \subset M$ by

$$\Psi : (q, s, r) \mapsto F_{\partial M}(\exp_q^{\partial M}(s\nu_P(q)), r).$$

Let $\hat{\Pi}$ be the map $(q, s, r) \mapsto (q, r)$, and let $\Pi : P \cap U_\Gamma \rightarrow \Gamma \times [0, \varrho_0)$ be defined by $\Pi = \hat{\Pi} \circ \Psi^{-1}$. Since P meets ∂M orthogonally along Γ , for all $q \in \Gamma$ we have $\Pi(q) = (q, 0)$, and it is easy to check that $D\Pi_q|_{T_q P}$ is an isomorphism. By the implicit function theorem, there is $\varrho > 0$ and an open set $U_\varrho \subset U_\Gamma$ such that for a neighbourhood Γ_p of p in Γ we have a smooth function $\psi : \Gamma_p \times [0, \varrho) \rightarrow (-\varrho, \varrho)$ with the property that

$$(4.2) \quad P \cap U_\varrho = \{\Psi(q, \psi(q, r), r) : q \in \Gamma_p, r \in [0, \varrho)\}, \quad \psi(q, 0) = 0, \quad \partial_r \psi(q, 0) = 0.$$

Define the function z_P on U_ϱ by $z_P(\Psi(q, s, r)) := s - \psi(q, r)$. We shall slightly abuse notation and write $z_P(q, s, r) = s - \psi(q, r)$. Notice that on $U_\varrho \cap \partial M$ (corresponding to $\{r = 0\}$) we have $\partial_r z_P = -\partial_r \psi(q, 0) = 0$, so $\langle \nabla z_P, \mathbf{n} \rangle = 0$ on $U_\varrho \cap \partial M$. Moreover, $\partial_s z_P = 1$ everywhere, so that $dz_P \neq 0$. Since $z_P = 0$ on $P \cap U_\varrho$ and $dz_P \neq 0$, then $\nabla^g z_P$ is normal to P on $P \cap U_\varrho$. Also, by construction, $|\nabla^g z_P| = 1$ on $U_\varrho \cap \partial M$, so that, by shrinking ϱ if necessary, we have $1/2 \leq |\nabla^g z_P| \leq 2$ on U_ϱ .

By the flowout theorem (see [Lee13, Theorem 9.20 and Theorem 9.34]), there is $\rho \in (0, \varrho)$ such that (v) in Definition 4.1 is satisfied with $U = U_\varrho$ and $R = \rho$. $\square$

Remark 4.4. For the compactness argument in Section 5.3, it suffices to consider reference half-discs obtained by restricting $F_{\partial M}(\Gamma \times [0, \varrho))$ to a neighbourhood of a point of Γ . In the coordinates of the preceding proof, these are given by $s = 0$, so that $\psi \equiv 0$ and $z_P = s$. By choosing $T_p \Gamma$ appropriately, such reference half-discs realize every tangent half-hyperplane at p orthogonal to ∂M . We nevertheless establish sheeting for arbitrary smooth two-sided reference half-discs meeting ∂M orthogonally, since the argument applies in this greater generality.

Assume $\rho_1 > 0$ is such that U_1 is an adapted ρ_1 -neighbourhood of $P_{\rho_1} \subset P$ as in Lemma 4.3. Thus, for $\delta \in (0, 2\rho_1]$, $R \in (0, 2\rho_1]$, we can define

$$(4.3) \quad \mathcal{N}_\delta^Z(P_R) := F_Z(P_R \times (-\delta, \delta)).$$

Let the inverse map F_Z^{-1} be given by $F_Z^{-1} = (\pi_Z, t_Z)$.

Remark 4.5. Notice that, since Z is tangential to ∂M along $\partial M \cap U_1$ and does not vanish anywhere, then the flow preserves ∂M and the interior of M , i.e. $F_Z(\partial_0 P_{2\rho_1} \times (-2\rho_1, 2\rho_1)) = \mathcal{N}_{2\rho_1}^Z(P_{2\rho_1}) \cap \partial M$.

Let $\Lambda \geq 0$ be such that

$$(4.4) \quad \begin{aligned} & \sup_{P_{2\rho_1}} |A_P|^2 + \sup_{P_{2\rho_1}} |\nabla_P A_P| + \sup_{P_{2\rho_1}} |\nabla_P^2 A_P|^{2/3} \leq \Lambda^2, \\ & \sup_{U_1} |\nabla_g^2 d_{\partial M}|^2 + \sup_{U_1} |\nabla_g^3 d_{\partial M}| + \sup_{U_1} |\nabla_g^4 d_{\partial M}|^{2/3} \leq \Lambda^2, \\ & \sup_{U_1} |\text{Rm}_g| + \sup_{U_1} |\nabla_g \text{Rm}_g|^{2/3} + \sup_{U_1} |\nabla_g^2 \text{Rm}_g|^{1/2} \leq \Lambda^2, \end{aligned}$$

where A_P denotes the second fundamental form of P in M .

Lemma 4.6. *Let P , ρ_1 , U_1 , z_P and Z be chosen as above, and suppose that (4.4) holds. There exists a constant $\delta_1 = \delta_1(n, \Lambda, \rho_1) > 0$ such that for all $\delta, R \in (0, \delta_1)$ the following estimates hold on $\mathcal{N}_{2\delta}^Z(P_{2R})$:*

$$(4.5) \quad |\nabla_g^2 z_P| \leq C(n)\Lambda, \quad |\nabla_g^3 z_P| \leq C(n)\Lambda^2,$$

$$(4.6) \quad |\nabla_g Z| \leq C(n)\Lambda, \quad |\nabla_g^2 Z| \leq C(n)\Lambda^2.$$

Moreover, the flow map F_Z restricted to $P_{2R} \times (-2\delta, 2\delta)$ is a diffeomorphism onto $\mathcal{N}_{2\delta}^Z(P_{2R})$ and it satisfies

$$(4.7) \quad |DF_Z| + |D(F_Z^{-1})| \leq C(n).$$

Finally, if $(y, t) \in P_{2R} \times (-2\delta, 2\delta)$ and $x = F_Z(y, t)$, then

$$(4.8) \quad \sup_{v \in T_y P, |v|=1} |\langle Z(x), (D_y F_Z)_{(y,t)} v \rangle| \leq C(n) (\sup |\nabla_g Z|) |t| \leq C(n)\Lambda |t|.$$

Proof. Recall the construction in the proof of Lemma 4.3, and the definition of the map $\Psi : \partial_0 P_{2\rho_1} \times (-2\rho_1, 2\rho_1) \times [0, 2\rho_1] \rightarrow U_1$ given by

$$\Psi : (q, s, r) \mapsto F_{\partial M}(\exp_q^{\partial M}(s\nu_P(q)), r).$$

Let $(\bar{g}_{ij})_{1 \leq i, j \leq n+1}$ denote the pullback metric tensor Ψ^*g and let $\theta = \theta(n, \Lambda, \rho_1) \in (0, 1/2)$ be a constant to be determined in the proof; then let $\rho_\theta := 2\theta\rho_1$. By the estimates of [LZ21b, Appendix A] and the Riccati equation, there exists a constant $c_0(n)$ such that, provided we choose $\theta > 0$ small enough that $\rho_\theta \Lambda \leq c_0(n)$, then

$$(4.9) \quad \frac{1}{2}g_{\text{Euc}} \leq \bar{g} \leq 2g_{\text{Euc}}, \quad |\Gamma(\bar{g})| \leq C(n)\Lambda, \quad |\partial\Gamma(\bar{g})| \leq C(n)\Lambda^2$$

on $\partial_0 P_{\rho_\theta} \times (-\rho_\theta, \rho_\theta) \times [0, \rho_\theta]$, where by $\Gamma(\bar{g})$ we denote the Christoffel symbols of the metric $\bar{g}$ and with $\partial\Gamma(\bar{g})$ the collection of their partial derivatives with respect to the coordinate vector fields. Moreover, since $D\psi = 0$ on $\partial_0 P_{\rho_1}$, by shrinking θ if needed, we can assume $|D\psi| \leq 1$ on $\partial_0 P_{\rho_\theta} \times [0, \rho_\theta]$. Therefore, using the graph representation in (4.2) together with the equation for the components of the second fundamental form of a graph, we obtain

$$(4.10) \quad |D^2\psi| \leq C(n) (\sup_{P_{2\rho_1}} |A_P| + |\Gamma(\bar{g})|) \leq C(n)\Lambda$$

on $\partial_0 P_{\rho_\theta} \times [0, \rho_\theta)$. Finally, differentiating the equation for the second fundamental form components in the tangential directions, we obtain

$$(4.11) \quad |D^3\psi| \leq C(n) (\sup_{P_{2\rho_1}} |\nabla_P A_P| + |\partial\Gamma(\bar{g})| + (\sup_{P_{2\rho_1}} |A_P| + |\Gamma(\bar{g})| + |D^2\psi|)^2) \leq C(n)\Lambda^2$$

after shrinking $c_0(n) > 0$ further, if necessary.

Recall in the proof of Lemma 4.3, we defined the function z_P by $z_P(\Psi(q, s, r)) := s - \psi(q, r)$. By differentiating the definition and using (4.10)–(4.11), it is immediate to obtain the higher order estimates (4.5) in $\mathcal{N}_{\rho_\theta}(P_{\rho_\theta})$. Finally, by definition of Z , for every vector X ,

$$(4.12) \quad \nabla_X^g Z = |\nabla^g z_P|^{-1} (\text{Id} - Z \otimes Z) (\nabla_g^2 z_P(X, \cdot))^\sharp.$$

Therefore, (4.6) follows from (4.12) and (4.5).

By standard ODE theory, bounds on $|Z|, |\nabla^g Z|, |\nabla_g^2 Z|$, together with (4.4), control the flowmap F_Z in C^2 . In particular, on $P_{\rho_\theta} \times (-\rho_\theta, \rho_\theta)$, after shrinking $c_0(n)$ if needed, since $|t| \leq \rho_\theta$ and $\rho_\theta \Lambda \leq c_0(n)$, we obtain (4.7) and (4.8). Thus, we can let $\delta_1 := \frac{1}{2}\theta\rho_1$, which satisfies all the required properties. $\square$

Now let us assume as in Theorem 1.14 that Σ is a two-sided free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset \mathcal{N}_\delta^Z(P_R)$ for some $\delta \in (0, \delta_1), R \in (0, R_1)$, where $R_1 \in (0, \delta_1)$ is to be specified later depending on n and Λ only.

Definition 4.7 (Adapted tilt function). Let $\mathcal{X} := \text{proj}_g^{T\Sigma}(Z)$, $\tau = |\mathcal{X}|_g$, and $\vartheta := g(\nu, Z)$, where Z is the vector field in Definition 4.1. Then τ is a tilt function on Σ , with respect to P , adapted to ∂M . We will refer to it, for brevity, as the *adapted tilt function*.

We shall now compute the differential inequality satisfied by τ , and notice that it is very similar to the inequality in Lemma 3.2.

Lemma 4.8. *Let P, δ_1, Z be as in Lemma 4.6. Additionally, let Σ be a smooth two-sided free-boundary minimal hypersurface in M and assume that $\Sigma \subset \mathcal{N}_\delta^Z(P_R)$. Let ν be a choice of unit normal to Σ , A its second fundamental form, and let $\tau, \mathcal{X}, \vartheta$ be as in Definition 4.7. Then $\tau^2 + \vartheta^2 = 1$. Let $\mathcal{B}$ be the $(0,2)$ -tensor on Σ corresponding to $\nabla^g Z|_{T\Sigma}$. Then the following hold.*

First, on $\{\tau > 0\}$, if $v := \mathcal{X}/\tau$, then

$$\nabla_\Sigma \tau = \vartheta A(\cdot, v) + \mathcal{B}(\cdot, v)$$

and, for every $\gamma > 0$,

$$|\nabla_\Sigma \tau|^2 \leq (1 + \gamma)(1 - \tau^2)|A(\cdot, v)|^2 + C(\gamma)|\mathcal{B}|^2.$$

Second, assuming that (4.4) holds, then for every $\gamma > 0$ on $\{\tau > 0\}$

$$\tau \Delta_\Sigma \tau \geq (1 - \gamma)|A|^2(1 - \tau^2) - \left(\frac{n-1}{n} + \gamma\right)|A|^2 - \gamma|A|^2\tau^2 - C(n, \gamma)\Lambda^2.$$

Proof. We shall follow the outline of the proof of Lemma 3.2, and highlight the analogies and differences. Notice that $|Z|_g = 1$ and $Z = \mathcal{X} + \vartheta\nu$ along Σ , so that $\tau^2 + \vartheta^2 = 1$.

Analogously to Lemma 3.2, let $\mathcal{B}$ be the $(0,2)$ -tensor on Σ corresponding to $\nabla^g Z|_{T\Sigma}$, so that for $V, W \in T\Sigma$, $\mathcal{B}(V, W)$ is defined by $\mathcal{B}(V, W) := \langle \nabla_V^g Z, W \rangle$.

Notice that for $V \in T\Sigma$

$$\nabla_V^g \mathcal{X} = \nabla_V^g Z - (V\vartheta)\nu - \vartheta \nabla_V^g \nu = \nabla_V^g Z - (V\vartheta)\nu + \vartheta A(V, \cdot)^\sharp.$$

If we take a tangential projection,

$$\nabla_V^\Sigma \mathcal{X} = (\nabla_V^g Z)^T + \vartheta A(V, \cdot)^\sharp.$$

Therefore for $V, W \in T\Sigma$,

$$(4.13) \quad g(\nabla_V^\Sigma \mathcal{X}, W) = g(\nabla_V^g Z, W) + \vartheta A(V, W) = \mathcal{B}(V, W) + \vartheta A(V, W).$$

By taking $W = \mathcal{X}$ in this identity, we immediately obtain that

$$\tau V\tau = \frac{1}{2}V(\tau^2) = g(\nabla_V^\Sigma \mathcal{X}, \mathcal{X}) = \mathcal{B}(V, \mathcal{X}) + \vartheta A(V, \mathcal{X}).$$

On $\{\tau > 0\}$, writing $\mathcal{X} = \tau v$, this becomes $V\tau = \vartheta A(V, v) + \mathcal{B}(V, v)$, or, equivalently, $\nabla_\Sigma \tau = \vartheta A(\cdot, v) + \mathcal{B}(\cdot, v)$. Taking norms, we obtain

$$|\nabla_\Sigma \mathcal{X}|^2 = \vartheta^2 |A|^2 + 2\vartheta \langle A, \mathcal{B} \rangle + |\mathcal{B}|^2 = |A|^2(1 - \tau^2) + 2\vartheta \langle A, \mathcal{B} \rangle + |\mathcal{B}|^2.$$

It follows from Young's inequality that for any fixed $\gamma > 0$

$$|\nabla_\Sigma \tau|^2 \leq (1 + \gamma)\vartheta^2 |A(\cdot, v)|^2 + C(\gamma)|\mathcal{B}(\cdot, v)|^2.$$

Since $\vartheta^2 = 1 - \tau^2$ and $|\mathcal{B}(\cdot, v)| \leq |\mathcal{B}|$, we obtain

$$|\nabla_\Sigma \tau|^2 \leq (1 + \gamma)(1 - \tau^2)|A(\cdot, v)|^2 + C(\gamma)|\mathcal{B}|^2.$$

Let $Y \in T\Sigma$ be a tangent vector field. Then we have the following Bochner identity (see [Pet16, Theorem 9.4.1])

$$\Delta_\Sigma \frac{|Y|^2}{2} = |\nabla_\Sigma Y|^2 + \langle Y, \nabla_\Sigma \operatorname{div}_\Sigma Y \rangle + \langle Y, \operatorname{div}_\Sigma ((\nabla_\Sigma Y) - (\nabla_\Sigma Y)^*) \rangle + \operatorname{Ric}_\Sigma(Y, Y).$$

As in the proof of Lemma 3.2, since $H = 0$, by the contracted Gauss equation

$$\operatorname{Ric}_\Sigma(Y, Y) = \operatorname{Ric}_g(Y, Y) - \operatorname{Rm}_g(Y, \nu, Y, \nu) - |A(Y, \cdot)|^2.$$

Now, letting $Y = \mathcal{X}$ in the Bochner formula, we notice that by (4.13),

$$\operatorname{div}_\Sigma \mathcal{X} = \operatorname{tr}_\Sigma(\mathcal{B} + \vartheta A) = \operatorname{tr}_\Sigma \mathcal{B} + \vartheta H = \operatorname{tr}_\Sigma \mathcal{B}$$

since $H = 0$ by minimality of Σ . Moreover, let $\mathcal{K}$ denote the skew-symmetric $(0, 2)$ -tensor on Σ defined by $\mathcal{K}(V, W) = \langle \nabla_V^\Sigma \mathcal{X}, W \rangle - \langle \nabla_W^\Sigma \mathcal{X}, V \rangle$ for $V, W \in T\Sigma$, and notice that this term does not appear in the proof of Lemma 3.2, since in that situation $\nabla_\Sigma X$ is symmetric, because X is a gradient vector field.

Since $\tau = |\mathcal{X}|_g$, we obtain

$$\begin{aligned} \tau \Delta_\Sigma \tau + |\nabla_\Sigma \tau|^2 &= \Delta_\Sigma \frac{\tau^2}{2} = |\nabla_\Sigma \mathcal{X}|^2 + \langle \mathcal{X}, \nabla_\Sigma \operatorname{tr}_\Sigma \mathcal{B} \rangle + \langle \mathcal{X}, \operatorname{div}_\Sigma \mathcal{K} \rangle + \operatorname{Ric}_\Sigma(\mathcal{X}, \mathcal{X}) \\ &= |A|^2(1 - \tau^2) + 2\vartheta \langle A, \mathcal{B} \rangle + |\mathcal{B}|^2 \\ &\quad + \langle \mathcal{X}, \nabla_\Sigma \operatorname{tr}_\Sigma \mathcal{B} \rangle + \langle \mathcal{X}, \operatorname{div}_\Sigma \mathcal{K} \rangle \\ &\quad + \operatorname{Ric}_g(\mathcal{X}, \mathcal{X}) - \operatorname{Rm}_g(\mathcal{X}, \nu, \mathcal{X}, \nu) - |A(\cdot, \mathcal{X})|^2, \end{aligned}$$

so that

$$(4.14) \quad \begin{aligned} \tau \Delta_\Sigma \tau &= |A|^2(1 - \tau^2) - |A(\mathcal{X}, \cdot)|^2 - |\nabla_\Sigma \tau|^2 + 2\vartheta \langle A, \mathcal{B} \rangle + |\mathcal{B}|^2 \\ &\quad + \langle \mathcal{X}, \nabla_\Sigma \operatorname{tr}_\Sigma \mathcal{B} \rangle + \langle \mathcal{X}, \operatorname{div}_\Sigma \mathcal{K} \rangle \\ &\quad + \operatorname{Ric}_g(\mathcal{X}, \mathcal{X}) - \operatorname{Rm}_g(\mathcal{X}, \nu, \mathcal{X}, \nu), \end{aligned}$$

Because $H = 0$, A is trace-free. Hence, for every unit vector v , $|A(v, \cdot)|^2 \leq \frac{n-1}{n}|A|^2$, and

$$|\nabla_\Sigma \tau|^2 \leq (1 + \gamma)(1 - \tau^2) \left(\frac{n-1}{n} \right) |A|^2 + C(\gamma)|\mathcal{B}|^2.$$

The same computations as in the proof of Lemma 3.2 also show that

$$|A(\cdot, \mathcal{X})|^2 + |\nabla_\Sigma \tau|^2 \leq \left(\frac{n-1}{n} + \gamma \right) |A|^2 + C(\gamma) |\mathcal{B}|^2$$

and

$$2|\vartheta \langle A, \mathcal{B} \rangle| \leq \gamma |A|^2 \vartheta^2 + C(\gamma) |\mathcal{B}|^2 = \gamma |A|^2 (1 - \tau^2) + C(\gamma) |\mathcal{B}|^2.$$

For the remaining terms in (4.14), similarly to Lemma 3.2, we have

$$|\nabla_\Sigma \operatorname{tr}_\Sigma \mathcal{B}| + |\operatorname{div}_\Sigma \mathcal{K}| \leq C(n) (|\nabla_g^2 Z| + |A| |\nabla_g Z|).$$

The bound in terms of Λ then follows from (4.4) since

$$|\operatorname{Ric}_g(\mathcal{X}, \mathcal{X}) - \operatorname{Rm}_g(\mathcal{X}, \nu, \mathcal{X}, \nu)| \leq C(n) |\operatorname{Rm}_g| \tau^2.$$

□

We also need to estimate the boundary normal derivative of the adapted tilt function along ∂M .

Lemma 4.9. *Let η be the inward unit conormal to $\partial \Sigma$ in Σ . Along $\partial \Sigma \cap \{\tau > 0\}$ we have $|\partial_\eta \tau| \leq |\mathbb{I}_{\partial M}| + |\nabla_g^g Z|$ pointwise.*

Proof. Let us differentiate the function ϑ in the direction of the unit conormal η to $\partial \Sigma$ along $\partial \Sigma \cap \{\tau > 0\}$. Notice that $\nabla_\eta^g \nu \in T\Sigma$, so that

$$\partial_\eta \vartheta = g(\nabla_\eta^g \nu, Z) + g(\nu, \nabla_\eta^g Z) = g(\nabla_\eta^g \nu, \mathcal{X}) + g(\nu, \nabla_\eta^g Z) = -A(\eta, \mathcal{X}) + g(\nu, \nabla_\eta^g Z).$$

By definition, Z is tangent to ∂M in $\partial M \cap \mathcal{N}_\delta^Z(P_R)$. Therefore, along $\partial \Sigma \cap \{\tau > 0\} \subset \partial M \cap \mathcal{N}_\delta^Z(P_R)$, we have

$$g(\mathcal{X}, \mathbf{n}) = g(Z - \vartheta \nu, \mathbf{n}) = g(Z, \mathbf{n}) - \vartheta g(\nu, \mathbf{n}) = 0,$$

and $\mathcal{X}$ is also tangent to ∂M along $\partial \Sigma$. Hence, by symmetry of the second fundamental form and the free-boundary condition

$$A(\eta, \mathcal{X}) = A(\mathcal{X}, \eta) = -g(\nabla_\mathcal{X}^g \nu, \eta) = -g(\nabla_\mathcal{X}^g \nu, \mathbf{n}) = -\mathbb{I}_{\partial M}(\mathcal{X}, \nu)$$

and

$$\partial_\eta \vartheta = \mathbb{I}_{\partial M}(\mathcal{X}, \nu) + g(\nu, \nabla_\eta^g Z).$$

Clearly, $|\mathbb{I}_{\partial M}(\mathcal{X}, \nu)| \leq |\mathbb{I}_{\partial M}| |\mathcal{X}| |\nu| = \tau |\mathbb{I}_{\partial M}|$. Moreover, since $|Z|_g \equiv 1$,

$$|g(\nu, \nabla_\eta^g Z)| = |g(\nu - \vartheta Z, \nabla_\eta^g Z)| \leq |\nu - \vartheta Z| |\eta| |\nabla_g^g Z| = \tau |\nabla_g^g Z|.$$

Finally,

$$\tau |\partial_\eta \tau| = |\vartheta \partial_\eta \vartheta| \leq |\vartheta| \tau (|\mathbb{I}_{\partial M}| + |\nabla_g^g Z|) \leq \tau (|\mathbb{I}_{\partial M}| + |\nabla_g^g Z|)$$

since $|\vartheta| \leq 1$. This concludes the proof. □

4.2. Caccioppoli-type inequality. We shall now move to the weak intrinsic Caccioppoli inequality. As in Section 3, we start with the smooth case, and then extend it to the singular set using the vanishing 2-capacity assumption on $\mathcal{S}$.

Lemma 4.10 (Caccioppoli inequality). *Let Σ be a stable singular free-boundary minimal hypersurface in M and assume that $\bar{\Sigma} \subset \mathcal{N}_\delta^Z(P_R)$. Assume that (4.4) holds. Then there exist a positive dimensional constant $C(n) > 0$ and a constant $K_{\text{Cac}} = K_{\text{Cac}}(n, \Lambda) > 0$ depending on n and Λ such that for every $k \in [0, 1/(4n)]$ and every $\varphi \in C_c^{0,1}(\mathcal{N}_{\rho_1}^Z(P_R))$, we have*

$$(4.15) \quad \frac{1}{2n} \int_{\{\tau > k\}} |A|^2 \left(1 - \frac{k}{\tau} \right) \varphi^2 \leq C(n) \left(\int_\Sigma (\tau - k)_+^2 |\nabla_\Sigma \varphi|^2 + K_{\text{Cac}}^2 \int_{\{\tau > k\}} \varphi^2 \right).$$

Consequently,

$$(4.16) \quad \frac{1}{2n} \int_{\{\tau > k\}} |\nabla_{\Sigma} \tau|^2 \left(1 - \frac{k}{\tau}\right) \varphi^2 \leq C(n) \left(\int_{\Sigma} (\tau - k)_+^2 |\nabla_{\Sigma} \varphi|^2 + K_{\text{Cac}}^2 \int_{\{\tau > k\}} \varphi^2 \right).$$

Proof. We shall first prove (4.15) and (4.16) for test functions $\varphi \in C_c^{0,1}(\Sigma)$, and then extend to compactly supported ambient Lipschitz functions by an excision argument, similar to the one in Lemma 2.14.

Let $\Omega_k := \{\tau > k\}$, and apply Lemma 2.16 to $\zeta = (\tau - k)_+ \varphi$ to obtain

$$(4.17) \quad \int_{\Sigma} |A|^2 (\tau - k)_+^2 \varphi^2 \leq (1 + \alpha) \int_{\Sigma} |\nabla_{\Sigma}((\tau - k)_+ \varphi)|^2 + C(n, \alpha) \Lambda^2 \int_{\Sigma} (\tau - k)_+^2 \varphi^2$$

Let $J := \int_{\Sigma} |\nabla_{\Sigma}((\tau - k)_+ \varphi)|^2$ and integrate by parts to get

$$(4.18) \quad \int_{\Sigma} |\nabla_{\Sigma}((\tau - k)_+ \varphi)|^2 = - \int_{\Omega_k} (\tau - k) \Delta_{\Sigma} \tau \varphi^2 + \int_{\Sigma} (\tau - k)_+^2 |\nabla_{\Sigma} \varphi|^2 - \int_{\partial \Sigma \cap \Omega_k} (\tau - k)_+ \partial_{\eta} \tau \varphi^2.$$

By Lemma 4.8, on Ω_k we have

$$(4.19) \quad -(\tau - k) \Delta_{\Sigma} \tau \leq - \left(1 - \frac{k}{\tau}\right) \left[\left((1 - \gamma)(1 - \tau^2) - \frac{n-1}{n} - \gamma - \gamma \tau^2 \right) |A|^2 - C(n, \gamma) \Lambda^2 \right].$$

Then (4.17) becomes

$$\begin{aligned} & \int_{\Omega_k} \left(1 - \frac{k}{\tau}\right) \left(\tau(\tau - k) + (1 + \alpha) \left(\frac{1}{n} - \tau^2 - 2\gamma \right) \right) |A|^2 \varphi^2 \\ & \leq (1 + \alpha) \int_{\Sigma} (\tau - k)_+^2 |\nabla_{\Sigma} \varphi|^2 + C(n, \alpha) \Lambda^2 \int_{\Sigma} (\tau - k)_+^2 \varphi^2 + (1 + \alpha) C(n, \gamma) \Lambda^2 \int_{\Omega_k} \varphi^2 \\ & \quad - (1 + \alpha) \int_{\partial \Sigma \cap \Omega_k} (\tau - k)_+ \partial_{\eta} \tau \varphi^2. \end{aligned}$$

The same choices $\alpha = \alpha_n, \gamma = \gamma_n$ (depending on n only) as in Lemma 3.4 make the coefficient in front of $(1 - k/\tau) |A|^2 \varphi^2$ bounded below by $1/(2n) + 1/(8n^2) > 1/(2n)$. Also, noticing that $0 \leq (\tau - k)_+ \leq 1$ and $(\tau - k)_+ = 0$ outside Ω_k , we can write the previous inequality as

$$\begin{aligned} \frac{1}{2n} \int_{\Omega_k} \left(1 - \frac{k}{\tau}\right) |A|^2 \varphi^2 & \leq C(n) \int_{\Sigma} (\tau - k)_+^2 |\nabla_{\Sigma} \varphi|^2 + C(n, \Lambda) \int_{\Omega_k} \varphi^2 \\ & \quad - C(n) \int_{\partial \Sigma \cap \Omega_k} (\tau - k)_+ \partial_{\eta} \tau \varphi^2. \end{aligned}$$

Let us now deal with the boundary term. Let $B_k := \int_{\partial \Sigma \cap \Omega_k} (\tau - k)_+ \partial_{\eta} \tau \varphi^2$. By Lemma 4.9,

$$|B_k| \leq \left(\sup_{U_1} |\mathbb{I}_{\partial M}| + \sup_{\mathcal{N}_{\delta}^Z(P_R)} |\nabla^g Z| \right) \int_{\partial \Sigma} (\tau - k)_+ \varphi^2 \leq C(n, \Lambda) \int_{\partial \Sigma} (\tau - k)_+ \varphi^2$$

where the second inequality follows from Lemma 4.6 and (4.4). By Lemma 2.14,

$$\int_{\partial \Sigma} (\tau - k)_+ \varphi^2 \leq \int_{\Omega_k} \varphi^2 |\nabla_{\Sigma} \tau| + 2 \int_{\Sigma} (\tau - k)_+ \varphi |\nabla_{\Sigma} \varphi| + C_{\text{tr}}(n, \Lambda) \int_{\Sigma} (\tau - k)_+ \varphi^2.$$

For every $\lambda > 0$, by the weighted Young inequality

$$\begin{aligned} C(n, \Lambda) \left(\int_{\Omega_k} \varphi^2 |\nabla_\Sigma \tau|^2 + 2 \int_\Sigma (\tau - k)_+ \varphi |\nabla_\Sigma \varphi|^2 \right) &\leq \lambda \int_{\Omega_k} \varphi^2 |\nabla_\Sigma \tau|^2 + \int_\Sigma (\tau - k)_+^2 |\nabla_\Sigma \varphi|^2 \\ &\quad + C(\lambda) C(n, \Lambda) \int_{\Omega_k} \varphi^2. \end{aligned}$$

Similarly, since $(\tau - k)_+ \leq \mathbb{1}_{\Omega_k}$, we can write $\int_\Sigma (\tau - k)_+ \varphi^2 \leq \int_{\Omega_k} \varphi^2$. Thus,

$$|B_k| \leq \lambda \int_{\Omega_k} \varphi^2 |\nabla_\Sigma \tau|^2 + \int_\Sigma (\tau - k)_+^2 |\nabla_\Sigma \varphi|^2 + C(\lambda) C(n, \Lambda) \int_{\Omega_k} \varphi^2.$$

Since $\varphi \nabla_\Sigma (\tau - k)_+ = \nabla_\Sigma ((\tau - k)_+ \varphi) - (\tau - k)_+ \nabla_\Sigma \varphi$, we have the obvious inequality $\int \varphi^2 |\nabla_\Sigma (\tau - k)_+|^2 \leq 2J + 2 \int (\tau - k)_+^2 |\nabla_\Sigma \varphi|^2$, so that

$$(4.20) \quad |B_k| \leq 2\lambda J + C(n, \lambda) \mathcal{R}_k$$

where $\mathcal{R}_k := \int (\tau - k)_+^2 |\nabla_\Sigma \varphi|^2 + C(n, \Lambda) \int_{\Omega_k} \varphi^2$.

Recall that by (4.18) and (4.19), we have

$$(4.21) \quad J \leq C(n) \int_{\Omega_k} \left(1 - \frac{k}{\tau} \right) |A|^2 \varphi^2 + C(n) \mathcal{R}_k + |B_k|.$$

Therefore, by combining (4.20) and (4.21), we obtain

$$|B_k| \leq 2\lambda \left(C(n) \int_{\Omega_k} \left(1 - \frac{k}{\tau} \right) |A|^2 \varphi^2 + C(n) \mathcal{R}_k + |B_k| \right) + C(n, \lambda) \mathcal{R}_k,$$

i.e.

$$(4.22) \quad |B_k| \leq \frac{2\lambda C(n)}{1 - 2\lambda} \int_{\Omega_k} \left(1 - \frac{k}{\tau} \right) |A|^2 \varphi^2 + \frac{2\lambda C(n) + C(n, \lambda)}{1 - 2\lambda} \mathcal{R}_k.$$

By choosing $\lambda > 0$ small enough (depending on n only) that $\frac{2\lambda C(n)}{1 - 2\lambda} < \frac{1}{8n^2}$, we can reabsorb the multiple of $\int_{\Omega_k} \left(1 - \frac{k}{\tau} \right) |A|^2 \varphi^2$ coming from (4.22) into the left hand side and obtain the desired inequality.

Finally, for ambient test function φ , first apply the inequality to $\chi_j \varphi$, where χ_j are the cutoff functions of Corollary B.6, chosen to equal one in the limit on $\text{spt } \varphi \setminus \mathcal{S}$, then take the limit as in the proof of Lemma 3.4. $\square$

4.3. Proof of Theorem 1.14. As in Section 3, we can now conclude the argument for Theorem 1.14 by applying the iteration lemma in Section 2.3.

Let us assume that (4.4) holds. Define $K_1(n, \Lambda) := \max\{K_{\text{Sob}}(n, \Lambda), K_{\text{Cac}}(n, \Lambda)\}$, let δ_1 be as in Lemma 4.6, and set

$$(4.23) \quad R_1 := \frac{1}{2} \min\{\delta_1, K_1^{-1}\}.$$

Let Σ be a stable singular free-boundary minimal hypersurface in M such that $\overline{\Sigma} \subset \mathcal{N}_\delta^Z(P_R)$ for some $\delta \in (0, \delta_1)$, $R \in (0, R_1)$.

Then, for $0 < r \leq R$ define the *adapted cylinder* $\mathcal{C}_r^Z$ by

$$(4.24) \quad \mathcal{C}_r^Z := F_Z(P_r \times (-2\delta, 2\delta)) = \mathcal{N}_{2\delta}^Z(P_r),$$

and consider the following *adapted tilt excess* quantity:

$$(4.25) \quad E_{\Sigma, P}^Z(r) := \left(r^{-n} \int_{\Sigma \cap \mathcal{C}_r^Z} \tau^2 \right)^{1/2}$$

where τ is the *adapted tilt function* in Definition 4.7.

Let us apply Lemma 2.18 with $u = \tau$ and $k_* = 1/(4n)$. Notice that Lemma 4.10 provides the necessary Caccioppoli-type inequality (2.5). Therefore, by Corollary 2.19, there exist $\varepsilon_1 = \varepsilon_1(n, \Lambda) > 0$ and $C_1 = C_1(n, \Lambda) > 0$ such that

$$\sup_{\Sigma \cap \mathcal{C}_{\sigma R}^Z} \tau \leq C_1 (1 - \sigma)^{-\frac{n}{n+2}} E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}}$$

provided $E_{\Sigma, P}^Z(R)^2 \leq \varepsilon_1^2 (1 - \sigma)^n$, which is the desired statement.

4.4. Proof of Theorem 1.15. The proof of the second sheeting theorem, Theorem 1.15, is rather similar to the proof of Theorem 1.12.

Assume $\bar{\Sigma} \subset \mathcal{N}_\delta^Z(P_R)$. By Theorem 1.14 with $\sigma = 3/4$, we have

$$(4.26) \quad \sup_{\Sigma \cap \mathcal{C}_{3R/4}^Z} \tau \leq C_1(n, \Lambda) E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}}.$$

By decreasing ε_1 if necessary, we may assume $\sup_{\Sigma \cap \mathcal{C}_{3R/4}^Z} \tau \leq \frac{1}{4}$.

Step 0: $(\Sigma \setminus \partial\Sigma) \cap \partial M \cap \mathcal{C}_{3R/4}^Z = \emptyset$. Let $\iota : N \rightarrow M \setminus \mathcal{S}$ parametrise Σ . If $z \in N \setminus \partial N$ and $x = \iota(z) \in \partial M \cap \mathcal{C}_{3R/4}^Z$, the function $d_{\partial M} \circ \iota$ has a local minimum at z , and $d(d_{\partial M} \circ \iota)_z = 0$, so that $T_x \Sigma \subset T_x \partial M$. By dimension counting, $T_x \Sigma = T_x \partial M$, which implies that $\tau(x) = 1$, contradicting (4.26).

Step 1: Local graphicality. Notice that

$$(4.27) \quad \text{Ker}(D\pi_Z)_x = \text{span}\{Z(x)\}$$

for every $x \in \mathcal{N}_\delta^Z(P_R)$. As in the proof of Theorem 1.12 in Section 3.4, if $x = \iota(z) \in \mathcal{C}_{3R/4}^Z$ for some $z \in N$, we can show that $D(\pi_Z \circ \iota)_z$ is an isomorphism. If $z \in \partial N$, the same argument and the identity $\pi_Z(\partial M) = P \cap \partial M$ show that in this case $\pi_Z \circ \iota$ is a local diffeomorphism of manifolds-with-boundary. In particular, similarly to Section 3.4, for every $z \in N$ such that $\iota(z) \in \mathcal{C}_{3R/4}^Z$ there is a (relatively open) neighbourhood (in N) the image of which is a smooth graph over a relatively open subset of $P_{3R/4}$ via the map F_Z .

Step 2: Finite covering away from the projected singular set. Let

$$\Gamma_Z := \pi_Z(\mathcal{S} \cap \overline{\mathcal{N}_\delta^Z(P_{3R/4})}) \subset \bar{P}_{3R/4}, \quad B_Z := P_{3R/4} \setminus \Gamma_Z.$$

As in the proof of Theorem 1.12, $\mathcal{H}^{n-1}(\Gamma_Z) = 0$. By the same argument as in the proof of Theorem 1.12, Γ_Z is relatively closed, and B_Z is open, dense and path-connected in $P_{3R/4}$. Define

$$N_Z := \{z \in N : \iota(z) \in \mathcal{C}_{3R/4}^Z, \pi_Z(\iota(z)) \in B_Z\},$$

and $\hat{\pi}_Z := \pi_Z \circ \iota|_{N_Z} : N_Z \rightarrow B_Z$. By *Step 1*, $\hat{\pi}_Z$ is a local diffeomorphism of manifolds-with-boundary and, by properness of ι as an immersion into $M \setminus \mathcal{S}$, it is also proper. Again, as in the proof of Theorem 1.12, it is a finite covering map of some degree Q .

Step 3: Lipschitz estimate and extension across Γ_Z . By *Step 2*, after ordering the preimages in the Z -fibre $F_Z(\{y\} \times (-\delta, \delta))$, we obtain locally Lipschitz functions $u_1^Z, \dots, u_Q^Z$ such that $u_1^Z(y) \leq$

$\cdots \leq u_Q^Z(y)$ for all $y \in B_Z$. Hence,

$$(4.28) \quad \Sigma \cap \pi_Z^{-1}(B_Z) = \cup_{j=1}^Q F_Z(\text{graph}(u_j^Z)).$$

For a smooth graph $G[w] = F_Z(\{(y, w(y)) : y \in P_R\})$ on P_R and $v \in T_y P$, where $x = F_Z(y, w(y))$, consider the vector

$$V(x) = (D_y F_Z)_{(y, w(y))}(v) + dw_y(v)Z(x) \in T_x G[w]$$

Since $|Z|_g = 1$, by taking the inner product with Z , we have

$$dw_y(v) = \langle V(x), Z(x) \rangle - \langle (D_y F_Z)_{(y, w(y))}(v), Z(x) \rangle.$$

Thus, by Lemma 4.6,

$$\begin{aligned} |dw_y(v)| &\leq |\text{proj}_g^{T_x G[w]} Z(x)| |V(x)| + |\langle (D_y F_Z)_{(y, w(y))}(v), Z(x) \rangle| \\ &\leq |\text{proj}_g^{T_x G[w]} Z(x)| (C(n)|v| + |dw_y(v)|) + C(n)\Lambda\delta|v|. \end{aligned}$$

If $|\text{proj}_g^{T_x G[w]} Z(x)| \leq 1/4$ (as is the case for our graphs u_j^Z , where this is just τ), then

$$|dw_y(v)| \leq C(n, \Lambda) \left(|\text{proj}_g^{T_x G[w]} Z(x)| + \delta \right) |v|.$$

Since we can approximate curves in $P_{3R/4}$ by curves which avoid Γ_Z , by (4.26) we conclude that

$$\text{Lip}_P(u_j^Z; B_Z) \leq C(n, \Lambda) \left(\delta + E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \right).$$

Similarly to the proof of Theorem 1.12, by density of B_Z in $P_{3R/4}$, every u_j^Z extends uniquely to a Lipschitz function (still denoted u_j^Z) on the whole of $P_{3R/4}$, with the same bound on the Lipschitz constant, namely

$$\text{Lip}_P(u_j^Z) \leq C(n, \Lambda) \left(\delta + E_{\Sigma, P}^Z(R)^{\frac{2}{n+2}} \right).$$

Finally, showing that $\bar{\Sigma} \cap \mathcal{N}_\delta^Z(P_{3R/4}) = \cup_{j=1}^Q F_Z(\text{graph}(u_j^Z))$ follows from the same argument as in the proof of Theorem 1.12.

As in Section 3, we can now prove (i) and (ii).

- (i) Assume $\mathcal{H}^{n-2}(\mathcal{S}) = 0$. Since π_Z is Lipschitz on $\mathcal{N}_{2\delta}^Z(P_{3R/4})$ by (4.7), then $\mathcal{H}^{n-2}(\Gamma_Z) = 0$. By reflecting across $\partial_0 P_R$ and applying the result in [SW16, Appendix A], we see that B_Z must then be simply connected, so every component of the covering $\hat{\pi}_Z : N_Z \rightarrow B_Z$ is one-sheeted. As in the proof of Theorem 1.12, this determines Q globally defined, not necessarily ordered Lipschitz functions $v_1^Z, \dots, v_Q^Z : B_Z \rightarrow (-\delta, \delta)$ such that

$$\Sigma \cap \pi_Z^{-1}(B_Z) = \cup_{j=1}^Q F_Z(\text{graph}(v_j^Z)),$$

each of which extends to a Lipschitz function (still denoted v_j^Z) on the whole of $P_{3R/4}$, preserving the Lipschitz constant.

As in the proof of Theorem 1.12, Γ_Z is relatively closed and $\mathcal{H}^{n-1}(\Gamma_Z) = 0$. Notice that the flow of Z preserves ∂M , so the vertical variation $v_j^Z + t\varphi$ is admissible on B_Z up to the boundary $\partial_0 P_{3R/4}$. Therefore, the Lipschitz functions v_j^Z are weak solutions of the minimal surface equation (with a uniform bound on the Lipschitz constant) for test functions compactly supported in B_Z . By direct computation, the area integrand $\mathcal{L}(y, z, q)$ in the local coordinates induced by the diffeomorphism Ψ (in Lemma 4.3) is uniformly elliptic by Lemma 4.6. Therefore, Lemma A.2 applies and the v_j^Z extend smoothly across Γ_Z all the way to the boundary, so that $\mathcal{S} \cap \mathcal{N}_\delta^Z(P_{R/2}) = \emptyset$, as required. The bound on $\|\nabla v_j^Z\|_{C^{1,\alpha}(P_{3R/4})}$ follows from standard elliptic regularity theory as in the proof of Theorem 1.12.

- (ii) Assume now Σ is embedded. Then for every $y \in B_Z$ the points of Σ lying in the Z -fibre are distinct and $u_1^Z < u_2^Z < \dots < u_Q^Z$ on B_Z , so that each u_j^Z is smooth on B_Z . The same removability argument as above (since $\mathcal{H}^{n-1}(\Gamma_Z) = 0$) implies that the u_j^Z are globally smooth, and weak ordering is preserved. Strict ordering is also preserved by the strong maximum principle for the minimal surface equation in the interior and by the corresponding boundary strong maximum principle (Hopf lemma) using the uniformly oblique boundary condition. Thus $\bar{\Sigma} \cap \mathcal{C}_{3R/4}^Z$ is a union of pairwise disjoint smooth graphs and $\mathcal{S} \cap \mathcal{N}_\delta^Z(P_{3R/4}) = \emptyset$. Finally, the gradient estimates follow from standard elliptic regularity theory as in (i).

5. COMPACTNESS

In this section we give a proof of Theorem 1.1. This roughly follows [SS81] and crucially uses Theorem 1.12 and Theorem 1.15, proven in the previous sections.

5.1. From height to tilt. We start by proving two propositions bounding the tilt excess by the L^2 and L^1 norms of the height. This will be used later in the proof of compactness to deduce from the varifold convergence smallness of the tilt, and consequently that we are in a regime in which we can apply the sheeting theorems. The Euclidean interior analogue is the standard tilt excess inequality (see e.g. [Bel25, Remark 4.2]).

We start with the boundary parallel case and then move to the orthogonal one.

Proposition 5.1 (Tilt-excess estimate). *Let M, P, ρ_0, R_0 be as in the statement of Theorem 1.11, and let $\delta \in (0, \rho_0), R \in (0, R_0)$. Suppose Σ is a singular properly immersed free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset \mathcal{N}_\delta^+(P_R)$. Then, there exists $C(n, \Lambda) > 0$ such that for $0 < 2r < R$ the following estimate holds*

$$(5.1) \quad \frac{1}{r^n} \int_{\Sigma \cap C_r} g^2 = \frac{1}{r^n} \int_{\Sigma \cap C_r} |\nabla_\Sigma d_{\partial M}|^2 \leq \frac{C(n, \Lambda)}{r^{n+2}} \int_{\Sigma \cap C_{2r}} (d_{\partial M}^2 + r^2 d_{\partial M}),$$

where g is the tilt function in Definition 3.1.

Proof. Consider the vector field $X(x) = d_{\partial M}(x) \varphi^2(x) \nabla^g d_{\partial M}(x)$, where $\varphi \in C_c^1(C_{2r})$ is a smooth cut-off taken to be identically 1 on C_r and satisfying $|\nabla \varphi| \leq C/r$ (for some absolute constant C). In particular, $X(x) \equiv 0$ on ∂M so that we can test the first variation with it. In particular, by free-boundary stationarity

$$0 = \int_\Sigma \operatorname{div}_\Sigma X = \int_\Sigma \varphi^2 g^2 + 2\varphi d_{\partial M} \langle \nabla_\Sigma \varphi, \nabla_\Sigma d_{\partial M} \rangle + \varphi^2 d_{\partial M} \operatorname{tr}_\Sigma \nabla_g^2 d_{\partial M},$$

since the mean curvature of Σ is assumed to vanish in $\Sigma \setminus \partial\Sigma$ by Definition 2.2. Hence,

$$\int_{\Sigma \cap C_r} g^2 \leq \int_\Sigma \varphi^2 g^2 \leq 2 \int_\Sigma |\varphi \nabla_\Sigma d_{\partial M}| |d_{\partial M} \nabla_\Sigma \varphi| + \sqrt{n} \int_\Sigma \varphi^2 d_{\partial M} |\nabla_g^2 d_{\partial M}|$$

from which (5.1) follows by Young's inequality and (1.2). $\square$

Proposition 5.2 (Orthogonal tilt-height excess estimate). *Let $M, P, \rho_1, \delta_1, R_1$ be as in the statement of Theorem 1.14. Let z_P and Z be as in Definition 4.1, and let $\delta \in (0, \delta_1), R \in (0, R_1)$. Suppose Σ is a singular properly immersed free-boundary minimal hypersurface in M such that $\bar{\Sigma} \subset \mathcal{N}_\delta^Z(P_R)$. Then, there exists $C(n, \Lambda) > 0$ such that for $0 < 2r < R$ the following estimate holds*

$$(5.2) \quad \frac{1}{r^n} \int_{\Sigma \cap C_r^Z} \tau^2 \leq \frac{C(n, \Lambda)}{r^{n+2}} \int_{\Sigma \cap C_{2r}^Z} (z_P^2 + r^2 |z_P|),$$

where τ is the adapted tilt function in Definition 4.7.

Moreover, since $|\nabla_{\Sigma} z_P|^2 = |\nabla^g z_P|^2 \tau^2$, the same estimate holds, up to increasing the constant C , with $|\nabla_{\Sigma} z_P|^2$ in place of τ^2 .

Proof. Let $W := |\nabla^g z_P|^{-1} Z$ and notice that $1/2 \leq |W|_g \leq 2$ by Definition 4.1. For $0 < 2r \leq R$ choose a cutoff χ on P_R which is compactly supported on P_{2r} and such that $0 \leq \chi \leq 1$, and $\chi \equiv 1$ on P_r , as well as $|\nabla_P \chi| \leq C/r$, where C is an absolute constant. Then pick a compactly supported vertical cutoff $\xi \geq \mathbb{1}_{[-\delta, \delta]}$ supported in $(-2\delta, 2\delta)$ and let $\varphi = (\xi \circ t_Z)(\chi \circ \pi_Z)$, so that $\varphi \equiv 1$ on $\mathcal{C}_r^Z$ as well as $\text{spt}(\varphi) \cap \Sigma \subset \mathcal{C}_{2r}^Z$. Also, φ satisfies $|\nabla \varphi| \leq C/r$, where C depends on n, Λ, ρ_1 only.

Consider then the vector field $Y = \varphi^2 z_P W$. Since W is tangent to ∂M along $\partial M \cap \mathcal{N}_\delta^Z(P_R)$, then Y is an admissible vector field for free-boundary stationarity and $\int_{\Sigma} \text{div}_{\Sigma} Y = 0$. Notice that

$$(5.3) \quad \langle \nabla_{\Sigma} z_P, W \rangle = \left\langle |\nabla^g z_P| Z^T, \frac{Z}{|\nabla^g z_P|} \right\rangle = |Z^T|^2 = \tau^2.$$

Therefore,

$$\begin{aligned} \text{div}_{\Sigma} Y &= \text{div}_{\Sigma}(\varphi^2 z_P W) = \varphi^2 \langle \nabla_{\Sigma} z_P, W \rangle + 2\varphi z_P \langle \nabla_{\Sigma} \varphi, W \rangle + \varphi^2 z_P \text{div}_{\Sigma} W \\ &= \varphi^2 \tau^2 + 2\varphi z_P |\nabla^g z_P|^{-1} \langle \nabla_{\Sigma} \varphi, Z^T \rangle + \varphi^2 z_P \text{div}_{\Sigma} W. \end{aligned}$$

Notice that

$$\text{div}_{\Sigma} W = |\nabla^g z_P|^{-1} \text{div}_{\Sigma} Z - \langle Z, |\nabla^g z_P|^{-2} \nabla_{\Sigma} |\nabla^g z_P| \rangle$$

so that $|\text{div}_{\Sigma} W| \leq C(n)(|\nabla^g Z| + |\nabla_{\Sigma}^2 z_P|) \leq C(n)\Lambda$ by Definition 4.1 and Lemma 4.6. Combining all of the above, we obtain

$$(5.4) \quad \int_{\Sigma} \varphi^2 \tau^2 \leq 4 \int_{\Sigma} \varphi |z_P| |\nabla_{\Sigma} \varphi| \tau + C(n)\Lambda \int_{\Sigma} \varphi^2 |z_P|.$$

By Young's inequality, $4\varphi |z_P| |\nabla_{\Sigma} \varphi| \tau \leq \frac{1}{2} \varphi^2 \tau^2 + 8z_P^2 |\nabla_{\Sigma} \varphi|^2$. After absorbing the first term into the left-hand side, it follows that

$$(5.5) \quad \int_{\Sigma} \varphi^2 \tau^2 \leq 16 \int_{\Sigma} z_P^2 |\nabla_{\Sigma} \varphi|^2 + C(n)\Lambda \int_{\Sigma} \varphi^2 |z_P|.$$

Using properties of φ , we conclude that

$$(5.6) \quad \int_{\Sigma \cap \mathcal{C}_r^Z} \tau^2 \leq \frac{C}{r^2} \int_{\Sigma \cap \mathcal{C}_{2r}^Z} z_P^2 + C(n, \Lambda) \int_{\Sigma \cap \mathcal{C}_{2r}^Z} |z_P|.$$

Dividing by r^n proves (5.2).

Finally, $|\nabla_{\Sigma} z_P|^2 = |(\nabla^g z_P)^T|^2 = |\nabla^g z_P|^2 |Z^T|^2 = |\nabla^g z_P|^2 \tau^2$. Thus, $(1/4)\tau^2 \leq |\nabla_{\Sigma} z_P|^2 \leq 4\tau^2$, which proves the corresponding estimate for $|\nabla_{\Sigma} z_P|^2$. $\square$

Remark 5.3. The height-tilt estimate also admits a formulation using only Z described in Remark 4.2, right after Definition 4.1. In that formulation, use the flow time t_Z as height. Since $Z(t_Z) = 1$ and $\nabla^g t_Z = Z$ on P , the derivative and flow estimates give

$$|\nabla^g t_Z - Z| \leq C(n)\Lambda |t_Z|.$$

Testing free-boundary stationarity with $\varphi^2 t_Z Z$ therefore yields (5.2) with t_Z in place of z_P : the error in replacing $\langle \nabla_{\Sigma} t_Z, Z \rangle$ by τ^2 is bounded by $C(n)\Lambda |t_Z| \tau$, and hence by the linear height term already present in the estimate.

5.2. Free-boundary stationary varifolds. Notice that Definition 2.6 implies that a singular properly embedded¹⁰ free-boundary minimal hypersurface Σ defines an integral varifold $V = |\Sigma| := \mathbf{v}(\Sigma, \mathbf{1}_\Sigma)$ on M . We refer to [Sim83] for the definition and basic properties of varifolds.

We define the *regular set* $\text{reg } V$ of such a varifold to be:

$$\text{reg}(V) := \left\{ x \in \text{spt } \|V\| : \begin{array}{l} \text{there exists } r > 0 \text{ such that } \text{spt } \|V\| \cap B_r(x) \text{ is a smooth} \\ \text{embedded hypersurface-with-(possibly empty) boundary in } \partial M \end{array} \right\}.$$

If $x \in \text{reg}(V)$ and $\text{spt } \|V\| \cap B_r(x)$ is a smooth embedded hypersurface-with-boundary Γ , then we say x is a *regular interior point* (denoted $x \in \text{reg}_i(V)$) if $x \in \Gamma \setminus \partial\Gamma$, and x is a *regular boundary point* if $x \in \partial\Gamma$ (denoted $x \in \text{reg}_b(V)$). Moreover, we define its complement to be the *singular set* $\text{sing}(V)$ of V by

$$\text{sing}(V) := \text{spt } \|V\| \setminus \text{reg}(V).$$

As we mentioned in Remark 2.8, given a singular properly embedded free-boundary minimal hypersurface Σ , its closure $\bar{\Sigma}$ in M , $\mathcal{S} = \bar{\Sigma} \setminus \Sigma$, and the corresponding multiplicity-one varifold $V = |\Sigma|$, we can redefine $\mathcal{S}$ and Σ so that

$$\text{reg}(V) = \Sigma, \quad \text{reg}_i(V) = \Sigma \setminus \partial\Sigma, \quad \text{reg}_b(V) = \partial\Sigma, \quad \text{sing}(V) = \mathcal{S}.$$

By (1)-(2) in Definition 2.2 and the discussion in Appendix B, the multiplicity-one varifold V associated to a singular properly embedded free-boundary minimal hypersurface Σ also satisfies an additional variational property, indeed such a V is *free-boundary stationary* in M . See [LPS24] for more details on free-boundary stationary varifolds. Some of the basic properties of free-boundary stationary varifolds which are needed in this paper are also briefly recalled in Appendix C.

For the multiplicity-one varifold V associated to a singular properly embedded free-boundary minimal hypersurface Σ , we can easily compute the first variation $\delta V(X)$ for *all* compactly supported vector fields $X \in \mathfrak{X}(M)$.

Lemma 5.4. *Let Σ be a singular properly embedded free-boundary minimal hypersurface in M and let $V = |\Sigma|$ be its associated multiplicity-one varifold. Then*

$$(5.7) \quad \delta V(X) = - \int_{\partial\Sigma} g(X, \eta) d\mathcal{H}^{n-1} \quad \text{for all } X \in \mathfrak{X}(M),$$

where η denotes the inner unit conormal of $\partial\Sigma$ in Σ . Hence, for all $K \Subset M$, if $K \Subset \text{int } K' \Subset M$, then

$$(5.8) \quad \sup \{ |\delta V(X)| : X \in \mathfrak{X}(M), |X|_g \leq 1, \text{spt } X \subset K \} \leq C(K) \mathcal{H}^n(\Sigma \cap K'),$$

i.e. the integral varifold $|\Sigma|$ has locally bounded first variation.

Proof. Notice that for all vector fields X supported away from the singular set $\mathcal{S}$, then (5.7) holds because of Definition 2.2. On the other hand, for a general $X \in \mathfrak{X}(M)$, if we let $\{\chi_j\}$ be the sequence of test functions in Corollary B.6, then $X_j := \chi_j X$ is supported away from the singular set, and the same excision argument as in the proof of Lemma 2.14 proves the general result. $\square$

Remark 5.5. Notice that our proof crucially uses the property that V is induced by a singular properly embedded free-boundary minimal hypersurface. On the other hand, (5.8) holds in general for free-boundary stationary varifolds, albeit with a more involved proof. The proof under the assumption of integrality is in [Ede20, Proposition 3.2], and in full generality in [DM21, Theorem 1.1].

¹⁰Most of the results in this specific subsection do actually hold for singular properly *immersed* free-boundary minimal hypersurfaces Σ with minimal changes in the statements and proofs.

The results that follow, although stated for varifolds induced by singular properly embedded free-boundary minimal hypersurfaces, hold for general free-boundary stationary integral varifolds and sequences thereof. See [LPS26] for a detailed treatment in this level of generality.

In this paper, we chose to restrict our treatment to varifolds induced by singular properly embedded free-boundary minimal hypersurfaces for the sake of simplicity and to offer a (more) self-contained presentation. Indeed, this special case will be sufficient for the applications in this paper, provided we supplement it with the following (elementary) observation.

Remark 5.6. The conclusions of Proposition 5.7, Lemma 5.8, Proposition 5.11, and Proposition C.7 also hold for varifold limits of sequences of singular properly embedded free-boundary minimal hypersurfaces in M with locally bounded mass. (This will be needed in Section 5.3 and Section 5.4).

By Allard's Compactness Theorem [All72, 6.4] and Lemma 5.8, we have the following.

Proposition 5.7. *Let (Σ_k) be a sequence of singular properly embedded free-boundary minimal hypersurfaces in M and, for each k , let $V_k = |\Sigma_k|$ be the associated multiplicity-one varifold. Assume that for each $K \Subset M$*

$$(5.9) \quad \sup_k \|V_k\|(K) \leq C(K) < +\infty.$$

Then, up to a subsequence (not relabelled), there is a free-boundary stationary integral varifold V_∞ on M such that

$$(5.10) \quad V_k \rightarrow V_\infty \quad \text{in the sense of varifolds.}$$

In particular, $\delta V_k \rightarrow \delta V_\infty$ and the limit has locally bounded first variation. Moreover,

$$(5.11) \quad \text{spt } \|V_k\| \rightarrow \text{spt } \|V_\infty\|$$

locally in Hausdorff distance.

Proof. The first variation hypothesis in [All72, 6.4] is fulfilled, in view of Lemma 5.4; this yields the subsequential limit V_∞ . The only conclusion that needs justification is (5.11), which follows from the standard proof for stationary integral varifolds (as remarked e.g. in [SS81, Remark 10]) once the following lemma is established. $\square$

Lemma 5.8. *Let Σ be a singular properly embedded free-boundary minimal hypersurface in M and let $V = |\Sigma|$ be the associated multiplicity-one varifold in M . Then for each compact $K \subset M$ there exist constants $c_K > 0, C_K, r_K > 0$ such that*

$$(5.12) \quad c_K r^n \leq \|V\|(B_r(y)) \leq C_K r^n$$

for all $y \in \text{spt } \|V\| \cap K$, $0 < r < r_K$.

Proof. Let $K \subset M$ be compact and pick $r_K > 0$ small enough that

- if $K \subset M \setminus \partial M$, $2r_K \leq \inf_{x \in K} \text{inj}_x(M)$;
- if $K \cap \partial M \neq \emptyset$, $F_{\partial M}$ is a diffeomorphism of $(K \cap \partial M) \times [0, 2r_K)$ onto the collar neighbourhood $\mathcal{N}_{2r_K}^+(K \cap \partial M)$ and $2r_K \leq \inf_{x \in K \setminus \mathcal{N}_{2r_K}^+(K \cap \partial M)} \text{inj}_x(M)$.

Then notice that by [Sim83, §17] and [GLZ20, Theorem 3.4] there exist constants $\Lambda_K^{\text{int}}, \Lambda_K^{\text{bd}}$ depending on $\sup_K |\text{Rm}_g|$ and $\sup_{\mathcal{N}_{2r_K}^+(K \cap \partial M)} |\nabla_g^2 d_{\partial M}|$ such that for all $y \in K$

$$(5.13) \quad e^{\Lambda_K^{\text{int}} \sigma} \frac{\|V\|(B_\sigma(y))}{\sigma^n} \leq e^{\Lambda_K^{\text{int}} \rho} \frac{\|V\|(B_\rho(y))}{\rho^n} \quad \text{whenever } B_\rho(y) \cap \partial M = \emptyset, 0 < \sigma < \rho < r_K,$$

and

$$(5.14) \quad e^{\Lambda_K^{\text{bd}} \sigma} \frac{\|V\|(B_\sigma(y))}{\sigma^n} \leq e^{\Lambda_K^{\text{bd}} \rho} \frac{\|V\|(B_\rho(y))}{\rho^n} \quad \text{whenever } y \in K \cap \partial M, 0 < \sigma < \rho < r_K.$$

For simplicity we shall prove the density ratio lower bound in (5.12). The proof of the upper bound follows a similar strategy. Choose $c_K = 16^{-n} \omega_n e^{-2 \max\{\Lambda_K^{\text{int}}, \Lambda_K^{\text{bd}}\} r_K} > 0$, and consider first a regular point $y \in \Sigma$. If $y \in \partial \Sigma$, then $\Theta(\|V\|, y) = 1/2$ and, by (5.14), we can conclude (5.12) holds for $0 < r < r_K$. Similarly if $y \in (\Sigma \setminus \partial \Sigma) \cap \partial M$.

Now fix $r \in (0, r_K)$. Since $y \in \Sigma \setminus \partial \Sigma$, then $\Theta(\|V\|, y) = 1$ and $d := \text{dist}_g(y, \partial M) > 0$. If $d \geq r/4$, then $B_{r/8}(y) \Subset M \setminus \partial M$ and (5.12) follows from (5.13). On the other hand, if $0 < d < r/4$, then let $\hat{y} = \pi_{\partial M}(y)$ be the nearest boundary point to y . By interior monotonicity at y at scale $d/2$, $\|V\|(B_{d/2}(y)) \geq c_K d^n$. Since $B_{d/2}(y) \subset B_{2d}(\hat{y})$, $\|V\|(B_{2d}(\hat{y})) \geq c_K (2d)^n$. By boundary monotonicity (5.14) from scale $2d$ to scale $r/2$,

$$\frac{\|V\|(B_{r/2}(\hat{y}))}{(r/2)^n} \geq c_K \frac{\|V\|(B_{2d}(\hat{y}))}{(2d)^n} \geq c_K$$

after absorbing exponential factors in the constant. Since $B_{r/2}(\hat{y}) \subset B_r(y)$, then (5.12) follows.

Finally, for a singular point $y \in \mathcal{S}$, choose regular points $\Sigma \ni y_\ell \rightarrow y$, and notice that for all sufficiently large ℓ , $B_{r/2}(y_\ell) \subset B_r(y)$, so (5.12) at y follows from the statement for regular points we have just proven. $\square$

Remark 5.9. A quantitative version of the density lower bound in Lemma 5.8 follows from [MS22, Theorem 9.22].

Remark 5.10. It is easy to see that the conclusion of Proposition 5.7 still holds if each Σ_k is a singular properly embedded free-boundary minimal hypersurface in (M, g_k) where the metrics g_k converge in C_{loc}^2 to a smooth metric g on M .

The notion of a *tangent cone* will be central to our analysis. We shall provide the definition and recap the basic properties of tangent cones to free-boundary stationary varifolds in Appendix C. For the purposes of the proof of Theorem 1.1, the following proposition will suffice.

Proposition 5.11. *Let Σ be a singular properly embedded free-boundary minimal hypersurface in M and let $V = |\Sigma|$ be the associated multiplicity-one varifold. If $x \in \text{spt } \|V\| = \bar{\Sigma} \subset M$, then $\text{VarTan}(V, x) \neq \emptyset$. Moreover,*

- (1) *if $x \in M \setminus \partial M$ and $\mathbf{C} \in \text{VarTan}(V, x)$, then $\mathbf{C}$ is a stationary integral cone in $(T_x M, g_{\text{Euc}})$ of density $\Theta(\|V\|, x)$;*
- (2) *if $x \in \partial M$ and $\mathbf{C} \in \text{VarTan}(V, x)$, then $\mathbf{C}$ is a free-boundary stationary integral cone in $(T_x^+ M, g_{\text{Euc}})$ of density $\Theta(\|V\|, x)$, and $\overline{\mathbf{C}} = \mathbf{C} + (\theta_0)_\# \mathbf{C}$ is a stationary integral cone in $(T_x M, g_{\text{Euc}})$ which is invariant under θ_0 , where θ_0 denotes the reflection across $T_x(\partial M) \subset T_x M$ in $(T_x M, g_{\text{Euc}})$.*

Finally, we shall need a version of Almgren's stratification for free-boundary stationary integral varifolds V . See Appendix C for the relevant notation and the statement of the relevant stratification result and [LPS26] for a proof.

We can now proceed with the proof of the Compactness Theorem (Theorem 1.1). Fix a compact $K \subset M$ and assume without loss of generality (by Proposition 5.7) that the multiplicity-one varifolds V_k associated with Σ_k converge in K to a free-boundary stationary varifold V . We then begin by analysing the possible tangent cones of V .

5.3. Top-dimensional stratum. We first analyse the top-dimensional part of the singular stratum of V , i.e. those (a priori, singular) points such that at least one tangent cone $\mathbf{C}$ is such that $\widehat{\mathbf{C}}$ has spine of dimension n .

Proposition 5.12 (Regularity at flat tangent cones). *Let $x \in \text{spt } \|V\|$ and assume $\mathbf{C} \in \text{VarTan}(V, x)$ is such that $\dim S(\widehat{\mathbf{C}}) = n$. Then x is a regular point of V and $\text{VarTan}(V, x) = \{\mathbf{C}\}$, i.e. the tangent cone at x is unique.*

Proof. For the sake of brevity, we shall focus on the case of $x \in \text{spt } \|V\| \cap \partial M$. The interior case is almost identical to (and, indeed, slightly simpler than) (1) in the dichotomy below. So, for the rest of this proof, let $x \in \text{spt } \|V\| \cap \partial M$, and, for simplicity, let $H = T_x M$, $H_+ = T_x^+ M$, and $T = \partial H_+ = T_x \partial M$.

By the constancy theorem [Sim83], $\overline{\mathbf{C}} = Q|\Pi|$ for some n -plane Π . By Proposition 5.11, the doubled cone $\overline{\mathbf{C}}$ is invariant under the reflection θ_0 across T . Hence, we know that the normal to Π is an eigenvector of the reflection map. Consequently, there are two options:

- (1) $\Pi = T_x \partial M$, in which case Q is even and $\mathbf{C} = (Q/2)|T_x \partial M|$.

Let the notation be as in Appendix C, so for $r > 0$ the map $\Psi_{x,r} : H_+ \cap B_{r^{-1}\rho_x}^H(0) \rightarrow M$ is the rescaling map in (C.2) with inverse $\Phi_{x,r}$, and $g_{x,r} = r^{-2}\Psi_{x,r}^*g$ as a metric on (a subset of) H_+ . By definition (see Appendix C.2 together with Remark 5.6), there exist positive numbers $r_i \rightarrow 0$ such that the free-boundary stationary integral varifolds $W_i := (\Phi_{x,r_i})_\# V$ in (H_+, g_i) converge locally to $(Q/2)|T| \subset H_+$, where for simplicity we let $g_i := g_{x,r_i}$. Let d_i denote the g_i -distance function to T in (H_+, g_i) . Moreover, for each i and $R > 0$, let P_R^i be the intrinsic ball in $(T, g_i|_T)$ of radius R centred at 0 and P_R the Euclidean ball in T . Since $g_i \rightarrow g_{\text{Euc}}$ locally smoothly, given any $\rho_0 > 0$, there exists $i_0 \geq 1$ such that

$$(5.15) \quad \begin{aligned} &\text{the normal exponential map of } T_x(\partial M) \text{ is a diffeomorphism of } P_{8\rho_0}^i \times [0, 8\rho_0) \text{ and} \\ &|\nabla_{g_i}^2 d_i|^2 + |\nabla_{g_i}^3 d_i| + |\text{Rm}_{g_i}| \leq 1 \text{ for all } i \geq i_0. \end{aligned}$$

By Proposition 5.7 (and Remark 5.6), $\text{spt } \|W_i\| \rightarrow T$ locally in Hausdorff distance. As $g_i \rightarrow g_{\text{Euc}}$, for any compact set $K \subset H_+$, we have

$$(5.16) \quad \sup_{\text{spt } \|W_i\| \cap K} d_i \rightarrow 0 \quad \text{as } i \rightarrow \infty^{11}.$$

Fix $\rho_0 \in (0, 1]$ and the corresponding $i_0 \geq 1$, and notice that by (5.15), the threshold radius R_0 and the threshold normalised tilt excess ε_0 in Theorem 1.12 can be chosen to hold uniformly for all $i \geq i_0$. By varifold convergence to $(Q/2)|T|$, we can assume that for all $i \geq i_0$ we have

$$(5.17) \quad \|W_i\|(\mathcal{N}_{4\rho_0}^+(P_{4\rho_0}^i)) \leq C(n, Q).$$

For fixed $R \in (0, R_0/2)$, by (5.17), we can choose $\delta \in (0, \rho_0/2)$ such that the right hand side of (5.1) with $r = 2R$ is strictly less than $\varepsilon_0^2/2$ for all $i \geq i_0$. By (5.16) with $K = \overline{\mathcal{N}_{2\rho_0}^+(P_{2R})}$, by possibly increasing i_0 , we can guarantee that for all $i \geq i_0$,

$$\text{spt } \|W_i\| \cap \mathcal{N}_{2\rho_0}^+(P_{2R}^i) \subset \mathcal{N}_{\delta/2}^+(P_{2R}^i)$$

¹¹Here we are using that since $g_i \rightarrow g_{\text{Euc}}$ locally smoothly, for any given compact set K , any given $\epsilon \in (0, 1)$ and all sufficiently large i the metrics satisfy $(1 - \epsilon)g_i \leq g_{\text{Euc}} \leq (1 + \epsilon)g_i$ on K . We shall use this fact repeatedly throughout the proof.

Now fix the scale $r_* = r_{i_0}$ and write $g_* := g_{x, r_{i_0}}$, $P_R^* := P_R^{i_0}$, $W^* = W_{i_0}$, and $W_k^* := (\Phi_{x, r_{i_0}})_\# V_k$. Notice that $W_k^* \rightarrow W^*$ locally as varifolds in the fixed Riemannian manifold-with-boundary (H_+, g_*) . By Proposition 5.7 (and Remark 5.6), for all sufficiently large k

$$\text{spt } \|W_k^*\| \cap \mathcal{N}_{2\rho_0}^+(P_{2R}^*) \subset \mathcal{N}_\delta^+(P_{2R}^*),$$

and W_k^* satisfies a mass bound analogous to (5.17).

Therefore, by Proposition 5.1, we obtain that, for all sufficiently large k , we have

$$E_{W_k^*, P^*}(2R)^2 \leq \varepsilon_0^2/2,$$

and Theorem 1.12 applies. In particular, (ii) of Theorem 1.12 applies and for all sufficiently large k (by uniform equivalence of the metrics)

$$(5.18) \quad \text{spt } \|W_k^*\| \cap \mathcal{N}_{2\rho_0}^+(P_{R/2}) = \bigcup_{j=1}^{Q_k} F_{\partial M}(\text{graph}(u_{k,j}))$$

for some $Q_k \in \mathbb{N}$ and smooth functions $0 \leq u_{k,1} < \dots < u_{k,Q_k}$, where $F_{\partial M}$ is the normal exponential map in the fixed rescaled metric. Moreover, Q_k is bounded above uniformly by some $\hat{Q}$ because of the uniform mass bound.

Finally, the gradient estimate in Theorem 1.12(ii) implies that, after passing to a subsequence, for each $j = 1, \dots, \hat{Q}$, the functions $u_{k,j}$ converge to some function u_j in $C^{2,\alpha}$ and $0 \leq u_1 \leq \dots \leq u_{\hat{Q}}$ on $P_{R/4}$. Each u_j satisfies the g_* -minimal surface equation on $P_{R/4}$, so if any two graphs touch they must coincide. After shrinking the neighbourhood further if needed, this shows that the support of W^* in a neighbourhood of 0 is given by a single smooth embedded minimal hypersurface, possibly with multiplicity, which must then be given by the density $Q/2$ at 0. After undoing all the rescalings, this shows that x is a regular point of V , and that the tangent cone is unique. It also gives local smooth graphical convergence of the hypersurfaces Σ_k .

- (2) $\Pi \subset T_x M$ is an n -plane invariant under reflection across $T_x \partial M$ and $\Pi \neq T_x \partial M$.

Notice that then $\mathbf{C} = Q|\Pi^+|$. Let $L = \Pi \cap T$ and notice that $\Pi = L \oplus \text{span}\{\mathbf{n}(x)\}$. For $\varrho \in (0, \text{inj}_x(\partial M))$, let $\Gamma := \exp_x^{\partial M}(L \cap B_\varrho(0))$, so that $T_x \Gamma = L$. Note that Γ is a smooth $(n-1)$ -dimensional embedded disc in ∂M . By shrinking ϱ if needed, the normal exponential map $F_{\partial M}$ is a diffeomorphism of $\Gamma \times [0, \varrho]$ into a smooth embedded two-sided n -dimensional disc P in M , with $\partial_0 P = \Gamma$. By construction, P meets ∂M orthogonally along Γ , and $T_y P = T_y \Gamma \oplus \text{span}\{\mathbf{n}(y)\}$ for all $y \in \Gamma$. At x , we have $T_x P = \Pi$. By Lemma 4.3 and Lemma 4.6, there is $\varrho_1 > 0$ such that P has an adapted ϱ_1 -neighbourhood, and the corresponding z_P, Z as in Definition 4.1 satisfy the estimates in Lemma 4.6 for some $\Lambda > 0$.

Now, as in (1), let $r_i \rightarrow 0$ be such that the free-boundary stationary integral varifolds $W_i := (\Phi_{x, r_i})_\# V$ in (H_+, g_i) converge locally to $Q|\Pi^+| \subset H_+$. Let $P^i := \Phi_{x, r_i}(P)$ and notice that P^i meets T orthogonally in (H_+, g_i) along $\partial_0 P^i$. Also, define $z_i := r_i^{-1} z_P \circ \Psi_{x, r_i}$, and let $Z_i := |\nabla^{g_i} z_i|_{g_i}^{-1} \nabla^{g_i} z_i$. It is easy to check that the flow of Z_i still defines an $r_i^{-1} \varrho_1$ -adapted neighbourhood U_i of P^i in (H_+, g_i) and that $P^i \rightarrow \Pi^+$, $z_i \rightarrow z_\Pi$, $Z_i \rightarrow \nu_\Pi$ locally smoothly, where ν_Π is a (Euclidean) unit normal to Π , and z_Π is the linear function vanishing on Π with gradient ν_Π . The corresponding flows also converge locally smoothly. Finally, for $R > 0$ and all i sufficiently large, let P_R^i be the intrinsic ball in $(P^i, g_i|_{P^i})$ of radius R centred at 0, which converges to the Euclidean ball P_R in Π^+ . Again, since $g_i \rightarrow g_{\text{Euc}}$ locally smoothly, there exists $i_1 \geq 1$ such that (4.4) holds with $\Lambda = 1$ for all $i \geq i_1$.

As in (1), by Proposition 5.7 (and Remark 5.6), $\text{spt } \|W_i\| \rightarrow \Pi^+$ locally in Hausdorff distance. As $g_i \rightarrow g_{\text{Euc}}$, for any compact set $K \subset H_+$, we have

$$(5.19) \quad \sup_{\text{spt } \|W_i\| \cap K} |z_i| \rightarrow 0 \quad \text{as } i \rightarrow \infty^{12}.$$

Fix $\rho_1 > 0$ and the corresponding $i_1 \geq 1$, and notice that the threshold distance δ_1 , the threshold radius R_1 and the threshold normalised tilt excess ε_1 in Theorem 1.15 can be chosen to hold uniformly for all $i \geq i_1$. Similarly to (1), by varifold convergence we have uniform mass bounds on any fixed compact set, so for a given $R \in (0, R_1/2)$, we can choose $\delta \in (0, \delta_1/2)$ such that the right hand side of (5.2) with $r = 2R$ is strictly less than $\varepsilon_1^2/2$ for all $i \geq i_1$.

Now the same argument as in (1) allows to fix a scale $r_* = r_{i_1}$ for which we can apply the orthogonal sheeting theorem (Theorem 1.15(ii)) to rescalings of the varifolds $V_k = |\Sigma_k|$ and deduce that for all large k , the support of the rescalings W_k^* in $\mathcal{N}_{2\rho_1}^{Z^*}(P_{R/2})$ is a union of Q smooth F_{Z^*} -graphs over $P_{R/2}$. The gradient estimates in Theorem 1.15(ii) show that the graphs converge in $C^{2,\alpha}$ to graphs whose union induces W^* on a smaller neighbourhood. A priori these graphs are weakly ordered but since they all satisfy a uniformly oblique boundary condition, the boundary and interior strong maximum principles for the minimal surface equation imply that any two graphs which touch must coincide. Therefore, by shrinking the neighbourhood further if needed, the support of W^* in a neighbourhood of 0 is given by a single smooth embedded free-boundary minimal hypersurface, possibly with multiplicity. Again as in (1), this shows that x is a regular point of V , and that the tangent cone is unique, together with local smooth graphical convergence of the Σ_k .

□

5.4. Excluding boundary classical cones: slicing argument. We can now move to the $(n-1)$ -dimensional part of the singular stratum. We will use the argument of [SS81]. Since the interior case is completely analogous to [SS81], we shall focus on the boundary case, i.e. $x \in \text{spt } \|V\| \cap \partial M$. Again, as in the proof of Proposition 5.12, we let $H = T_x M$, $H_+ = T_x^+ M$, and $T = \partial H_+ = T_x \partial M$.

Before we proceed with the argument we need the following elementary lemma, which shows that if $\mathbf{C}$ is a tangent cone of a stable singular properly embedded free-boundary minimal hypersurface Σ at some $x \in \bar{\Sigma} = \text{spt } |\Sigma|$, then the regular part of $\mathbf{C}$ is a stable free-boundary minimal cone (for the Euclidean metric on $T_x M$). Similarly for tangent cones to varifold limits of stable singular properly embedded free-boundary minimal hypersurfaces.

Lemma 5.13. *Let V be the multiplicity-one varifold induced by a stable singular properly embedded free-boundary minimal hypersurface Σ , or the varifold limit of a sequence of such objects. Let $x \in \text{spt } \|V\|$ and let $\mathbf{C} \in \text{VarTan}(V, x)$. Then $\text{reg}(\mathbf{C})$ is a stable smooth (possibly free-boundary) minimal hypersurface in $(T_x M, g_{\text{Euc}})$.*

Proof. Let the notation be as in Appendix C and the proof of Proposition 5.12. For simplicity, let us assume $V = |\Sigma|$ for a stable singular properly embedded free-boundary minimal hypersurface $|\Sigma|$ and $x \in \bar{\Sigma} \cap \partial M$. By definition, there exists a sequence $r_i \rightarrow 0$ such that the multiplicity-one varifolds $|\Sigma_i|$ induced by stable singular free-boundary minimal hypersurfaces $\Sigma_i = \Phi_{x, r_i}(\Sigma)$ in the rescaled metrics g_i converge locally to $\mathbf{C}$. By tracking how the geometric quantities involved

¹²Again we are using uniform equivalence to g_{Euc} , in addition to the properties of z_i .

transform, we see that, for every $\varphi \in C_c^{0,1}(\Sigma_i)$, we have

$$(5.20) \quad \begin{aligned} & \int_{\Sigma_i} |\nabla_{\Sigma_i}^{\mathbf{g}_i} \varphi|_{\mathbf{g}_i}^2 - r_i^2 \left((|A_{\Sigma}|_{\mathbf{g}}^2 + \text{Ric}_{\mathbf{g}}(\nu_{\Sigma}, \nu_{\Sigma})) \circ \Psi_{x, r_i} \right) \varphi^2 d\mathcal{H}_{\mathbf{g}_i}^n \\ & - r_i \int_{\partial \Sigma_i} (\mathbb{I}_{\partial M}^{\mathbf{g}}(\nu_{\Sigma}, \nu_{\Sigma}) \circ \Psi_{x, r_i}) \varphi^2 d\mathcal{H}_{\mathbf{g}_i}^{n-1} \geq 0. \end{aligned}$$

By Corollary B.6), then (5.20) holds for all compactly supported ambient Lipschitz test functions.

Now let $y \in \text{reg}(\mathbf{C})$. Then any tangent cone $\mathbf{D}$ to $\mathbf{C}$ at y is a constant multiplicity hyperplane or orthogonal half-plane. Therefore, after applying Proposition 5.12 to a diagonal sequence converging to $\mathbf{D}$, we see that the free-boundary minimal hypersurfaces in the diagonal sequence converge smoothly and graphically to $\mathbf{D}$ on a neighbourhood of y . In particular, we can pass (5.20) to the limit in a neighbourhood of y and obtain that $\text{reg}(\mathbf{C})$ satisfies the (Euclidean) stability inequality

$$(5.21) \quad \int_{\text{reg}(\mathbf{C})} |A_{\mathbf{C}}|^2 \varphi^2 \leq \int_{\text{reg}(\mathbf{C})} |\nabla \varphi|^2$$

for all compactly supported Lipschitz test functions φ .

Finally, the case where V is a varifold limit of stable singular properly embedded free-boundary minimal hypersurfaces follows from the same proof after taking a further diagonal subsequence. $\square$

We are now ready to prove the main result in this subsection, namely that for a varifold limit V of a sequence of stable singular properly embedded free-boundary minimal hypersurfaces the $(n-1)$ -dimensional part of the singular stratum $\mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2}$ is empty. Another way to formulate this is that if $\mathbf{C}$ is a tangent cone to such a varifold V and $\dim_{\mathcal{H}} S(\widehat{\mathbf{C}}) \geq n-1$, then actually $\dim_{\mathcal{H}} S(\widehat{\mathbf{C}}) = n$ and Proposition 5.12 applies.

Proposition 5.14. *Let V be a varifold limit of a sequence $\{\Sigma_k\}_k$ of stable singular properly embedded free-boundary minimal hypersurfaces in M . Then the $(n-1)$ -dimensional part of the singular stratum $\mathcal{S}_{n-1} \setminus \mathcal{S}_{n-2}$ of V is empty.*

Proof. As we stated above, we shall focus on the case where $x \in \text{spt} \|V\| \cap \partial M$. As in the proof of Proposition 5.12, we let $H = T_x M$, $H_+ = T_x^+ M$, $T = \partial H_+$, and $H_+^{\circ} = H_+ \setminus T$.

Let $\mathbf{C} \in \text{VarTan}(V, x)$ and assume that $\overline{\mathbf{C}}$ has an $(n-1)$ -dimensional spine, denoted by L . Since $\overline{\mathbf{C}}$ is invariant under reflection θ_0 across $T = \partial H_+$, then L is invariant under θ_0 . As the (-1) -eigenspace of the reflection is one-dimensional, exactly one of the following occurs: either $L \subset T$, or $\mathbf{n}_x \in L$, where $\mathbf{n}_x$ is the inward unit normal to T .

If $\mathbf{n}_x \in L$, choose a nonzero point $z \in H_+^{\circ} \cap L$. In a sufficiently small ball centred at z , the reflected part of the cone lies outside H_+ , and hence $\mathbf{C}$ agrees with the full cone $\overline{\mathbf{C}}$ there. Since $\overline{\mathbf{C}}$ is invariant along L , its tangent cone at z is again a classical cone with $(n-1)$ -dimensional spine. The usual interior slicing argument of [SS81] gives the contradiction. Hence, it remains to treat the genuinely free-boundary case $L \subset T$.

Choose coordinates on $H \cong \mathbb{R}^{n+1}$ so that $H_+ = \{x_1 \geq 0\}$ and $T = \partial H_+ = \{x_1 = 0\}$. After rotation, assume that L is given by $\{x_1 = x_{n+1} = 0\} = \{0\} \times \mathbb{R}^{n-1} \times \{0\} \subset T$. Write a point as $X = (x_1, y, x_{n+1})$, where $y = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}$. For $\alpha \in [0, \pi]$, define the n -dimensional half-hyperplane

$$H_{\alpha}^+ = \{(r \sin(\alpha), y, -r \cos(\alpha)) : r \geq 0, y \in \mathbb{R}^{n-1}\}.$$

Then $\partial H_{\alpha}^+ = L$ for every α . Furthermore, $H_0^+, H_{\pi}^+ \subset T$ (actually $T = H_0^+ \cup H_{\pi}^+$), while $H_{\pi/2}^+ = \{x_{n+1} = 0, x_1 \geq 0\}$ meets T orthogonally. Set $\omega_{\alpha} = \sin(\alpha)e_1 - \cos(\alpha)e_{n+1}$, so that ω_{α} points from L into H_{α}^+ . A choice of unit normal to H_{α}^+ is given by $\nu_{\alpha} = \cos(\alpha)e_1 + \sin(\alpha)e_{n+1}$, while the outward

conormal of H_α^+ along L is $\eta_\alpha = -\omega_\alpha = -\sin(\alpha)e_1 + \cos(\alpha)e_{n+1}$. Consequently, since the cone is invariant along L , it has the form

$$\mathbf{C} = q_0|H_0^+| + \sum_{i=1}^m p_i|H_{\alpha_i}^+| + q_\pi|H_\pi^+|,$$

with $0 < \alpha_1 < \alpha_2 < \dots < \pi$, $p_i \in \mathbb{N}$, $q_0, q_\pi \in \mathbb{N} \cup \{0\}$. Because $\mathbf{C}$ is only free-boundary, we do not have the full vector balancing equation. However, testing the first variation near L with a vector field equal to e_{n+1} which is tangent to T , gives the identity $q_0 - q_\pi + \sum_{i=1}^m p_i \cos(\alpha_i) = 0$. There is no corresponding balancing condition for the e_1 -component.

We first prove that $\alpha_i = \pi/2$ for every i . Choose a diagonal sequence realising the tangent cone $\mathbf{C}$. Thus, after relabelling, $\Sigma_j = \Phi_{x,r_j}(\Sigma_{k_j})$ and $V_j = (\Phi_{x,r_j})_\# V_{k_j}$, then $V_j \rightarrow \mathbf{C}$ locally as varifolds. We also have the corresponding sequence of Riemannian metrics as in Proposition 5.12. Every point of $\text{spt} \|\mathbf{C}\| \setminus L$ has a planar tangent cone. On the one hand, at points of $H_{\alpha_i}^+ \setminus L$, with $0 < \alpha_i < \pi$, we are in the interior of H , and hence interior sheeting [SS81] applies. On the other hand, at points of $(H_0^+ \cup H_\pi^+) \setminus L$, boundary sheeting (cf. Theorem 1.12) applies. Consequently, $\Sigma_j \rightarrow \mathbf{C}$ smoothly and graphically on compact subsets of $\overline{H}_+ \setminus L$. In other words, on every fixed annulus around the spine L , the graphical sheets, their tangent planes, and their unoriented normals converge (locally) uniformly to those of $\mathbf{C}$.

For $\tau > 0$, define the following τ -tubular neighbourhood $\mathcal{L}_\tau$ of the spine by

$$\mathcal{L}_\tau = \{(x_1, y, x_{n+1}) \in B_{3/2}^+ : x_1^2 + x_{n+1}^2 < \tau^2, |y| < 1\}$$

Choose $\zeta \in C_c^1(B_2)$ such that $0 \leq \zeta \leq 1$, $\zeta \equiv 1$ on $B_{3/2}$, and $|D\zeta| \leq 3$. Since the metrics are converging locally in C^2 , applying the stability inequality (5.20) for Σ_j , reabsorbing the boundary terms using Lemma 2.14, and then using uniform equivalence of g_j to g_{Euc} gives a uniform bound $\int_{\Sigma_j \cap B_{3/2}^+} |A_j|^2 \leq C$.

Furthermore, cover $L \cap \{|y| < 1\}$ by at most $C\tau^{1-n}$ balls of radius 2τ with centres on $L \subset T$. Boundary-centred monotonicity and the uniform mass bound give mass at most $C\tau^n$ in each corresponding ball of radius 4τ . Hence $\mathcal{H}_{g_j}^n(\Sigma_j \cap \mathcal{L}_\tau) \leq C\tau$, uniformly for all sufficiently large j and $0 < \tau < \tau_0$. Therefore, we obtain

$$(5.22) \quad \int_{\Sigma_j \cap \mathcal{L}_\tau} |A_j| \leq \left(\int_{\Sigma_j \cap B_{3/2}^+} |A_j|^2 \right)^{1/2} \left(\mathcal{H}_{g_j}^n(\Sigma_j \cap \mathcal{L}_\tau) \right)^{1/2} \leq C\tau^{1/2}.$$

Let now $\pi : H_+ \rightarrow L$, be given by $\pi(x_1, y, x_{n+1}) = y$. For $y \in \mathbb{R}^{n-1}$, define $P_y^+ := \pi^{-1}(y) = \{(x_1, y, x_{n+1}) : x_1 \geq 0\}$ and $D_{\tau,y}^+ = \{(x_1, y, x_{n+1}) \in P_y^+ : x_1^2 + x_{n+1}^2 \leq \tau^2\}$. We denote the curved boundary of this half-disk by $\Lambda_{\tau,y} = \{(x_1, y, x_{n+1}) : x_1 \geq 0, x_1^2 + x_{n+1}^2 = \tau^2\}$. Since $\mathcal{S}_j$ has locally finite $\mathcal{H}^{n-2}$ measure, we have $\mathcal{H}^{n-1}(\mathcal{S}_j) = 0$, and hence, since π is Lipschitz, $\mathcal{H}^{n-1}(\pi(\mathcal{S}_j)) = 0$. Consequently, for almost every $y \in B_{1/2}^{n-1}$, the point y does not belong to $\pi(\mathcal{S}_j)$ and, by Sard's theorem, is a regular value of both $\pi|_{\Sigma_j \setminus \mathcal{S}_j}$ and $\pi|_{\partial\Sigma_j}$. Additionally, the intersection with $\Lambda_{\tau,y}$ is transverse for all sufficiently large j . For all such y , the slice $\Gamma_{j,y} := \Sigma_j \cap D_{\tau,y}^+$ is a one-dimensional, smooth manifold with $\partial\Gamma_{j,y} \subset (\Sigma_j \cap \Lambda_{\tau,y}) \cup (\partial\Sigma_j \cap D_{\tau,y}^+)$. Notice that a point of $(\Sigma_j \setminus \partial\Sigma_j) \cap T$ is an interior point of $\Gamma_{j,y}$ rather than an endpoint. In particular, every connected component of $\Gamma_{j,y}$ is either a smooth embedded circle, or a smooth embedded arc with exactly two endpoints. In the latter case, the unordered pair of endpoint types is one of $\{\Lambda_{\tau,y}, \Lambda_{\tau,y}\}$, $\{\Lambda_{\tau,y}, \partial\Sigma_j\}$, or $\{\partial\Sigma_j, \partial\Sigma_j\}$. No other case arises: the slice does not meet $\mathcal{S}_j$ or the curved part of $\partial B_{3/2}^+$, while properness prevents connected components from terminating or accumulating in the interior of $D_{\tau,y}^+$.

Let N_j denote a continuous choice of Euclidean unit normal to Σ_j , and regard $[N_j]: \Sigma_j \rightarrow \mathbb{RP}^n$ as the unoriented Gauss map. For a curve $\gamma \subset \Sigma_j$, write

$$\text{osc}_\gamma[N_j] = \sup_{a,b \in \gamma} d_{\mathbb{RP}^n}([N_j(a)], [N_j(b)]).$$

Suppose for the sake of contradiction that $\alpha_{i_0} \neq \pi/2$ for some i_0 . For every regular slice, choose one of the outer endpoints $a_{j,y} \in \Sigma_j \cap \Lambda_{\tau,y}$ corresponding to the ray $H_{\alpha_{i_0}}^+$, and let $\gamma_{j,y} \subset \Gamma_{j,y}$ be the connected component containing it. Since $\gamma_{j,y}$ has an endpoint, it is an arc. Denote its other endpoint by $b_{j,y}$. Let $\mathcal{A}_\mathbf{C} := \{\alpha_1, \dots, \alpha_m\} \cup \{0 : q_0 > 0\} \cup \{\pi : q_\pi > 0\}$ and define

$$(5.23) \quad \delta_\mathbf{C} := \min \left\{ d_{\mathbb{RP}^n}([\nu_{\alpha_{i_0}}], \mathbb{P}(e_1^\perp)), d_{\mathbb{RP}^n}([\nu_{\alpha_{i_0}}], [\nu_\alpha]) : \alpha \in \mathcal{A}_\mathbf{C}, \alpha \neq \alpha_{i_0} \right\},$$

where the latter family is omitted if empty. Since $d_{\mathbb{RP}^n}([\nu_{\alpha_{i_0}}], \mathbb{P}(e_1^\perp)) = |\alpha_{i_0} - \pi/2| > 0$, and we have finitely many rays, $\delta_\mathbf{C} > 0$. We now have three cases for $b_{j,y}$:

- (i) If $b_{j,y} \in \partial\Sigma_j$, then the free-boundary condition gives $d_{\mathbb{RP}^n}([N_j(b_{j,y})], \mathbb{P}(e_1^\perp)) = o_j(1)$, while smooth convergence at the outer endpoint gives $[N_j(a_{j,y})] \rightarrow [\nu_{\alpha_{i_0}}]$. Hence, for all sufficiently large j , $\text{osc}_{\gamma_{j,y}}[N_j] \geq \delta_\mathbf{C}/2$.
- (ii) If $b_{j,y} \in \Lambda_{\tau,y}$ corresponds to a ray H_α^+ with $\alpha \neq \alpha_{i_0}$, then $[N_j(a_{j,y})] \rightarrow [\nu_{\alpha_{i_0}}]$, and $[N_j(b_{j,y})] \rightarrow [\nu_\alpha]$. Therefore, for all sufficiently large j , $\text{osc}_{\gamma_{j,y}}[N_j] \geq \delta_\mathbf{C}/2$.
- (iii) Suppose finally that $b_{j,y} \in \Lambda_{\tau,y}$ corresponds to the same ray $H_{\alpha_{i_0}}^+$. Set $E = \text{span}\{e_1, e_{n+1}\}$, and let N_j be as above. Since y is a regular value of $\pi|_{\Sigma_j}$, we have $T_p\gamma_{j,y} = \ker D(\pi|_{\Sigma_j})_p = T_p\Sigma_j \cap E$, and $\dim T_p\gamma_{j,y} = 1$. In particular, $\text{proj}_E N_j \neq 0$ along $\gamma_{j,y}$: otherwise $N_j \in E^\perp$, and hence $E \subset N_j^\perp = T_p\Sigma_j$, contradicting the fact that the kernel above is one-dimensional. Thus, $\widehat{N}_{j,y} := |\text{proj}_E N_j|^{-1} \text{proj}_E N_j$ is a unit normal to the planar curve $\gamma_{j,y}$. Orient this curve from $a_{j,y}$ to $b_{j,y}$, and let $T_{j,y}$ denote its Euclidean unit tangent. Smooth convergence at the two outer endpoints gives, for all sufficiently large j , $\langle T_{j,y}(a_{j,y}), \omega_{\alpha_{i_0}} \rangle < -1/2$, and $\langle T_{j,y}(b_{j,y}), \omega_{\alpha_{i_0}} \rangle > 1/2$, since the curve enters $D_{\tau,y}^+$ at the first endpoint and leaves it at the second. If $\text{osc}_{\gamma_{j,y}}[N_j]$ were sufficiently small, then $[\widehat{N}_{j,y}]$ would remain close to $[\nu_{\alpha_{i_0}}]$, and hence the tangent line would remain close to $[\omega_{\alpha_{i_0}}]$. In particular, for some fixed $c > 0$, $|\langle T_{j,y}, \omega_{\alpha_{i_0}} \rangle| \geq c$ along $\gamma_{j,y}$. This contradicts the intermediate value theorem, since the scalar product is negative at $a_{j,y}$ and positive at $b_{j,y}$. Therefore, $\text{osc}_{\gamma_{j,y}}[N_j] \geq \delta_{\text{turn}}$, for some $\delta_{\text{turn}} > 0$ independent of j, y, τ .

Taking $\beta := \min\{\delta_\mathbf{C}/2, \delta_{\text{turn}}\} > 0$, we conclude that $\text{osc}_{\gamma_{j,y}}[N_j] \geq \beta$ for all sufficiently large j and almost every regular $y \in B_{1/2}^{n-1}$.

Next, there exists a sequence $\varepsilon_j \rightarrow 0$ such that, along every g_j -unit speed curve in Σ_j , there holds $|\partial_s N_j| \leq C(|A_j| + \varepsilon_j)$. Thus, we obtain

$$\beta \leq C \int_{\gamma_{j,y}} (|A_j| + \varepsilon_j) d\mathcal{H}_{g_j}^1, \quad \text{and} \quad \beta \leq C \int_{\Sigma_j \cap P_y^+ \cap \mathcal{L}_\tau} (|A_j| + \varepsilon_j) d\mathcal{H}_{g_j}^1,$$

for almost every $y \in B_{1/2}^{n-1}$. Integrating and using coarea, we obtain

$$\beta \mathcal{L}^{n-1}(B_{1/2}^{n-1}) \leq C \int_{B_{1/2}^{n-1}} \int_{\Sigma_j \cap P_y^+ \cap \mathcal{L}_\tau} (|A_j| + \varepsilon_j) d\mathcal{H}_{g_j}^1 dy \leq C \int_{\Sigma_j \cap \mathcal{L}_\tau} |A_j| d\mathcal{H}_{g_j}^n + C\varepsilon_j \mathcal{H}_{g_j}^n(\Sigma_j \cap \mathcal{L}_\tau),$$

where we have used the uniform bound $J_{n-1}(\pi|_{\Sigma_j}) \leq C$. We conclude that $\beta \mathcal{L}^{n-1}(B_{1/2}^{n-1}) \leq C\tau^{1/2} + C\varepsilon_j\tau$. First let $j \rightarrow \infty$, with $\tau > 0$ fixed. We obtain $\beta \mathcal{L}^{n-1}(B_{1/2}^{n-1}) \leq C\tau^{1/2}$. Letting $\tau \downarrow 0$, the

right-hand side tends to zero, whereas the left-hand side is strictly positive. This contradiction proves that $\alpha_i = \pi/2$ for every i .

We have proven that $\mathbf{C} = q_0|H_0^+| + p|H_{\pi/2}^+| + q_\pi|H_\pi^+|$, where $p = \sum_i p_i$. Appealing to the free-boundary balancing equation, we infer $q_0 - q_\pi = 0$. Consequently, $q_0 = q_\pi = q$ and $q|H_0^+| + q|H_\pi^+| = q|T|$. Thus, $\mathbf{C} = p|H_{\pi/2}^+| + q|T|$.

A completely analogous argument allows to treat the mixed case $p, q \geq 0$. More precisely, one can show that $pq = 0$, in which case either $\mathbf{C} = q|T|$, or $\mathbf{C} = p|H_{\pi/2}^+|$. Doubling and appealing to Theorem 1.12, or Theorem 1.15, we conclude. $\square$

We can finally conclude the proof of Theorem 1.1 by a simple and (by now) rather standard argument using dimension reduction and Simons' classification of stable regular cones in low dimensions.

5.5. Conclusion. The key result is the following spine-increase result, which allows the reduction to the aforementioned Simons' classification of stable regular cones in low dimensions. Let $\mathbf{D}$ be a cone and let $S(\mathbf{D})$ be its spine. If $y \in \text{spt}\|\mathbf{D}\|$ and $\mathbf{E} \in \text{VarTan}(\mathbf{D}, y)$, then $S(\mathbf{D}) + \text{span}\{y\} \subset S(\mathbf{E})$. In particular, if $y \notin S(\mathbf{D})$, then $\dim S(\mathbf{E}) \geq \dim S(\mathbf{D}) + 1$. Using this and appealing to Simons' classification of stable regular cones of codimension one, we conclude that $\dim_{\mathcal{H}} \mathcal{S} \leq n - 7$. In particular, when $n \leq 6$, $\mathcal{S} = \emptyset$, while in the case $n = 7$, we can additionally prove that $\mathcal{S}$ is discrete. For the sake of contradiction, assume there are distinct singular points $z_j \in \mathcal{S}$ converging to z , and set $\rho_j = d_g(z_j, z)$. Blow-up V around z at scale ρ_j to obtain, after passing to a subsequence, a tangent cone $\mathbf{C}$. Furthermore, the rescaled points $\rho_j^{-1} \exp_z^{-1}(z_j)$ converge to a point $z^* \in S^n$ satisfying $z^* \in \text{sing}(\mathbf{C})$. Since $\mathbf{C}$ is a cone, $\{tz^*: t > 0\} \subset \text{sing}(\mathbf{C})$ which is impossible since we have established that for $n = 7$, $\mathcal{H}^\gamma(\mathcal{S}) = 0$ for all $\gamma > 0$.

5.6. Stability of the limit. As in [SS81, Section 5], the previous steps have already established the dimension bound on the singular set $\mathcal{S}$ and local smooth graphical convergence to Σ on $M \setminus \mathcal{S}$.

Notice that by stationarity in the interior and the constancy theorem, the density of the limit varifold is constant on each connected regular component.

Now let Γ be a connected component of Σ . For $x \in \Gamma$, $y \in \partial\Gamma \subset \partial M$ define

$$a_\Gamma(x) := |A_\Gamma(x)|^2 + \text{Ric}_M(\nu(x), \nu(x)), \quad b_\Gamma(y) := \mathbb{I}_{\partial M}(\nu(y), \nu(y)),$$

where $\nu(x)$ is any choice of unit normal to Γ at x , and similarly for $\nu(y)$. Notice that a_Γ , b_Γ are well-defined on Γ and $\partial\Gamma$ respectively, since each term is invariant under $\nu \mapsto -\nu$. Now define

$$(5.24) \quad Q_\Gamma(\varphi, \varphi) = \int_\Gamma (|\nabla_\Gamma \varphi|^2 - a_\Gamma \varphi^2) - \int_{\partial\Gamma} b_\Gamma \varphi^2$$

for $\varphi \in C_c^{0,1}(\Gamma)$.

Let $\varphi \in C_c^{0,1}(\Gamma)$ be arbitrary. Since Γ is embedded, there is a relatively open neighbourhood $U \Subset M \setminus \mathcal{S}$ of the support of φ which does not intersect any other component of Σ , and an extension $\hat{\varphi} \in C_c^{0,1}(U)$ of φ to U . By (iv) in Theorem 1.1, if $\varphi_k := \hat{\varphi}|_{\Sigma_k}$, then

$$(5.25) \quad \int_{\Sigma_k} |\nabla_{\Sigma_k} \varphi_k|^2 - \int_{\Sigma_k} (|A_{\Sigma_k}|^2 + \text{Ric}_M(\nu_{\Sigma_k}, \nu_{\Sigma_k})) \varphi_k^2 - \int_{\partial\Sigma_k} \mathbb{I}_{\partial M}(\nu_{\Sigma_k}, \nu_{\Sigma_k}) \varphi_k^2 \geq 0$$

for all k . As $|\Sigma_k| \rightarrow \theta|\Sigma|$ in the sense of varifolds, and Σ_k converges locally smoothly to Σ on $M \setminus \mathcal{S}$, then, after taking the limit as $k \rightarrow \infty$ in (5.25), we obtain that $mQ_\Gamma(\varphi, \varphi) \geq 0$, where m is the positive constant $\theta|_\Gamma$. Therefore $Q_\Gamma(\varphi, \varphi) \geq 0$.

For the one-sided case, we adapt the even-odd argument in the proof of [Bel22, Lemma A.1] to the free-boundary setting. If Γ is two-sided, the condition $Q_\Gamma(\varphi, \varphi) \geq 0$ for all $\varphi \in C_c^{0,1}(\Gamma)$ says

exactly that the stability inequality (1.1) is satisfied with Γ in place of Σ_k and therefore that Γ is stable.

Now suppose that Γ is one-sided, and consider the orientable double cover $\tilde{\Gamma}$ of Γ in the normal bundle $N\Gamma$ defined by

$$\tilde{\Gamma} = \{(x, v) : x \in \Gamma, v \in N_x\Gamma, |v|_g = 1\}.$$

Equip $\tilde{\Gamma}$ with the induced pullback metric. The immersion $\tilde{\iota} : \tilde{\Gamma} \rightarrow M$ given by $\tilde{\iota}(x, v) = x$ is free-boundary minimal and has globally defined unit normal $\tilde{\nu}_{\tilde{\Gamma}}(x, v) = v$. Let

$$Q_{\tilde{\Gamma}}(\phi, \phi) := \int_{\tilde{\Gamma}} |\nabla_{\tilde{\Gamma}} \phi|^2 - \int_{\tilde{\Gamma}} (|A_{\tilde{\Gamma}}|^2 + \text{Ric}_M(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}})) \phi^2 - \int_{\partial \tilde{\Gamma}} \mathbb{I}_{\partial M}(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}}) \phi^2$$

for all $\phi \in C_c^{0,1}(\tilde{\Gamma})$.

If $\phi \in C_c^{0,1}(\tilde{\Gamma})$ is *even*, i.e. $\phi(x, -v) = \phi(x, v)$, then it descends to a function $\varphi \in C_c^{0,1}(\Gamma)$ defined by $\varphi(x) = \phi(x, v)$ for either unit normal $v \in N_x\Gamma$, and, by definition,

$$Q_{\tilde{\Gamma}}(\phi, \phi) = 2Q_{\Gamma}(\varphi, \varphi) \geq 0.$$

If $\phi \in C_c^{0,1}(\tilde{\Gamma})$ is *odd*, i.e. $\phi(x, -v) = -\phi(x, v)$, then $|\phi|$ is an even compactly supported Lipschitz function on $\tilde{\Gamma}$ such that $|\nabla_{\tilde{\Gamma}} |\phi||^2 = |\nabla_{\tilde{\Gamma}} \phi|^2$ almost everywhere. Then

$$Q_{\tilde{\Gamma}}(\phi, \phi) = Q_{\tilde{\Gamma}}(|\phi|, |\phi|) \geq 0.$$

Finally, an arbitrary $\phi \in C_c^{0,1}(\tilde{\Gamma})$ decomposes into the sum $\phi = \phi_e + \phi_o$ of an even and an odd function and it is easy to check that by definition of $Q_{\tilde{\Gamma}}$, we have $Q_{\tilde{\Gamma}}(\phi_e, \phi_o) = 0$. Therefore, for every $\phi \in C_c^{0,1}(\tilde{\Gamma})$

$$Q_{\tilde{\Gamma}}(\phi, \phi) = Q_{\tilde{\Gamma}}(\phi_e, \phi_e) + Q_{\tilde{\Gamma}}(\phi_o, \phi_o) \geq 0.$$

Therefore,

$$\int_{\tilde{\Gamma}} |\nabla_{\tilde{\Gamma}} \phi|^2 - \int_{\tilde{\Gamma}} (|A_{\tilde{\Gamma}}|^2 + \text{Ric}_M(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}})) \phi^2 - \int_{\partial \tilde{\Gamma}} \mathbb{I}_{\partial M}(\tilde{\nu}_{\tilde{\Gamma}}, \tilde{\nu}_{\tilde{\Gamma}}) \phi^2 \geq 0,$$

for all $\phi \in C_c^{0,1}(\tilde{\Gamma})$.

APPENDIX A. REMOVABILITY OF SINGULARITIES

We now prove the simple PDE result that is needed in the proofs of the sheeting theorems, namely Theorem 1.12 and Theorem 1.15. The result is somewhat standard, and we state it here in the exact form we need in the paper.

Let $D \subset \mathbb{R}^n$ be a bounded open domain and let E be a (possibly empty) relatively open smooth subset of ∂D .

Lemma A.1. *Let Γ be a relatively closed subset of $D \cup E$ and assume $\mathcal{H}^{n-1}(\Gamma) = 0$. For every compact set $K \subset D \cup E$ there exist $\chi_j \in C^\infty(D \cup E)$ with $0 \leq \chi_j \leq 1$, each χ_j vanishing on a relative neighbourhood of $\Gamma \cap K$ in $D \cup E$ such that $\chi_j \rightarrow 1$ locally on $(D \cup E) \cap K \setminus \Gamma$ and $\int_{D \cap K} |\nabla \chi_j| \rightarrow 0$.*

Proof. This follows from [EG15, Theorem 4.15(v)]. $\square$

Let $C_0, C_1 > 0$ be fixed and let $\mathcal{L} : (D \cup E) \times (-C_0, C_0) \times B_{C_1}^n(0) \rightarrow \mathbb{R}$ be a smooth function. Assume there are constants $0 < \lambda_1 \leq \lambda_2 < +\infty$ such that for every $(y, z, q) \in (D \cup E) \times (-C_0, C_0) \times B_{C_1}^n(0)$ and $\xi \in \mathbb{R}^n$

$$(A.1) \quad \lambda_1 |\xi|^2 \leq \partial_{q_i q_j} \mathcal{L}(y, z, q) \xi^i \xi^j \leq \lambda_2 |\xi|^2.$$

Lemma A.2. *Let Γ be a relatively closed subset of $D \cup E$ and assume $\mathcal{H}^{n-1}(\Gamma) = 0$. Let u be a Lipschitz function on $D \cup E$ such that*

$$(A.2) \quad \|u\|_{L^\infty(D \cup E)} \leq c_0 < C_0, \quad \text{Lip}(u; D \cup E) \leq c_1 < C_1,$$

and suppose that the identity

$$(A.3) \quad \int_D (D_q \mathcal{L}(y, u, \nabla u) \cdot \nabla \varphi + \partial_z \mathcal{L}(y, u, \nabla u) \varphi) dy = 0$$

holds for every test function $\varphi \in C^1(D \cup E)$ such that $\text{spt } \varphi \Subset (D \cup E) \setminus \Gamma$.

Then (A.3) holds for every $\varphi \in C^1(D \cup E)$ such that $\text{spt } \varphi \Subset (D \cup E)$ and $u \in C_{\text{loc}}^\infty(D \cup E)$. In particular, u is a classical solution of

$$(A.4) \quad -\partial_{y_i}(\partial_{q_i} \mathcal{L}(y, u, \nabla u)) + \partial_z \mathcal{L}(y, u, \nabla u) = 0$$

and

$$(A.5) \quad D_q \mathcal{L}(y, u, \nabla u) \cdot \eta_D = 0 \quad \text{on } E$$

where η_D is the outward unit normal to D along E .

Proof. Fix $\varphi \in C^1(D \cup E)$ such that $\text{spt } \varphi \Subset (D \cup E)$ and let $K = \text{spt } \varphi$. Then let χ_j be the cutoff functions provided by Lemma A.1. Then $\chi_j \varphi$ is an admissible test function for (A.3), so that

$$(A.6) \quad 0 = \int_D \chi_j (D_q \mathcal{L}(y, u, \nabla u) \cdot \nabla \varphi + \partial_z \mathcal{L}(y, u, \nabla u) \varphi) dy + \int_D \varphi D_q \mathcal{L}(y, u, \nabla u) \cdot \nabla \chi_j dy$$

By (A.2) and smoothness of $\mathcal{L}$, there is some large constant $C < +\infty$ such that

$$\|D_q \mathcal{L}(y, u, \nabla u)\|_{L^\infty(D \cap K)} + \|\partial_z \mathcal{L}(y, u, \nabla u)\|_{L^\infty(D \cap K)} \leq C < +\infty.$$

Hence,

$$\left| \int_D \varphi D_q \mathcal{L}(y, u, \nabla u) \cdot \nabla \chi_j dy \right| \leq C \|\varphi\|_{L^\infty} \int_{D \cap K} |\nabla \chi_j| \rightarrow 0$$

by Lemma A.1. The first integral in (A.6) converges by the dominated convergence theorem. Therefore (A.3) holds for φ , and, hence, by arbitrariness of φ for all compactly supported C^1 functions on $D \cup E$.

Smoothness of u in the interior then follows from elliptic regularity theory since the divergence form operator associated to (A.3) is uniformly elliptic by (A.1), the coefficients of $\mathcal{L}$ are assumed to be smooth and u is Lipschitz. Along E , the natural boundary operator is $\mathcal{B}(y, z, q) := D_q \mathcal{L}(y, z, q) \cdot \eta_D(y)$. Notice that it is uniformly oblique as

$$D_q \mathcal{B}(y, z, q) \cdot \eta_D = \partial_{q_i q_j} \mathcal{L}(y, z, q) \eta_D^i \eta_D^j \geq \lambda_1 |\eta_D|^2 = \lambda_1 > 0.$$

By [Lie88, Theorem 1], there exists $\alpha \in (0, 1)$ such that $u \in C_{\text{loc}}^{1, \alpha}(D \cup E)$. By standard elliptic theory, smoothness follows from $C^{2, \alpha}$ regularity, which is a consequence of the boundary Schauder estimates of [LT86]. $\square$

Remark A.3. Notice that Lemma A.2 is considerably weaker and more elementary than Simon's removable singularity theorem in [Sim77]. Indeed, in Lemma A.2 the function u is already defined and Lipschitz across the deleted set Γ , so only the weak Euler–Lagrange identity must be extended. Simon on the other hand assumes no a priori boundedness or continuity of the solution near Γ and proves the existence of a classical extension. However, Simon's theorem as formulated in [Sim77] does not cover integrands having z -dependence or oblique boundary conditions.

APPENDIX B. CAPACITY AND EXTENSION OF STATIONARITY

We elaborate on Remark 2.7. In particular, we prove that under the assumptions of Definition 2.6 we can extend stationarity across $\mathcal{S}$. The proof uses some ideas introduced by Harvey–Lawson in [HL14] and employs some tools from classical potential theory. The case $n = 2$ of Proposition B.5 is due to Simon [Sim82] (without assuming local finiteness a priori).

The following lemma provides the basic analytic construction behind Proposition B.5, and is based on [HL14, Proposition 10.2-10.4]. For the sake of completeness, we provide a detailed proof.

Lemma B.1. *Let $n \geq 2$, and let $E \subset \mathbb{R}^{n+1}$ be compact with $\mathcal{H}^{n-2}(E) < \infty$. Then there exists an upper semicontinuous function $\psi : \mathbb{R}^{n+1} \rightarrow [-\infty, \infty)$ such that*

$$\{\psi = -\infty\} = E, \quad \psi \in C^\infty(\mathbb{R}^{n+1} \setminus E),$$

and

$$\mathrm{tr}_P D^2 \psi(x) \geq 0 \quad \text{for every } x \in \mathbb{R}^{n+1} \setminus E \text{ and } P \in G(n, n+1).$$

Proof. Let K be the fundamental solution of the Laplacian in $\mathbb{R}^n$, so that $K(r) \sim r^{2-n}$ for $n \geq 3$ and $K(r) \sim -\log r$ for $n = 2$. We first consider $n \geq 3$. Define $U_{n-2}^\sigma(x) := \int K(|x-y|) d\sigma(y)$ for a finite positive measure σ . We shall use the following elementary fact (see [Fal03, Theorem 4.13(a)]).

Fact B.2. Let $F \subset \mathbb{R}^{n+1}$ be Borel, and let σ be a nonzero finite positive measure supported on F . If

$$I_{n-2}(\sigma) := \iint K(|x-y|) d\sigma(x) d\sigma(y) < \infty,$$

then $\mathcal{H}^{n-2}(F) = \infty$.

Let now $\lambda := \mathcal{H}^{n-2} \llcorner E$, which is a finite positive measure by hypothesis. Set $Z := \{x \in E : U_{n-2}^\lambda(x) < \infty\}$, and $Z_m := \{x \in E : U_{n-2}^\lambda(x) \leq m\}$, so that $Z_m \subset Z$ and $Z = \bigcup_m Z_m$. We claim that $\mathcal{H}^{n-2}(Z) = 0$.

Indeed, if $\lambda(Z_m) > 0$ for some m , then the nonzero measure $\sigma := \lambda \llcorner Z_m$ satisfies

$$I_{n-2}(\sigma) = \int_{Z_m} U_{n-2}^\sigma d\lambda \leq \int_{Z_m} U_{n-2}^\lambda d\lambda \leq m\lambda(Z_m) < \infty,$$

and Fact B.2 would then imply $\mathcal{H}^{n-2}(Z_m) = \infty$, contradicting $Z_m \subset E$ and $\mathcal{H}^{n-2}(E) < \infty$. Hence $\lambda(Z_m) = 0$ for every m , and therefore $\mathcal{H}^{n-2}(Z) = \lambda(Z) = 0$.

Thus $U_{n-2}^\lambda(\cdot) = +\infty$ at every point of $E \setminus Z$. To conclude the argument, we shall now construct another finite positive measure ν such that $U_{n-2}^\nu(x) = +\infty$ for all $x \in Z$ as well. In the end, we shall define $\mu := \lambda + \nu$ and take $\psi = -U_{n-2}^\mu$, which, as we will show, has the desired properties.

Since $\mathcal{H}^{n-2}(Z) = 0$, for each $k \geq 1$ we may choose a countable cover

$$Z \subset \bigcup_i B_{r_{k,i}}(a_{k,i}), \quad a_{k,i} \in Z,$$

such that $\sum_i r_{k,i}^{n-2} < 2^{-k}$. Define $\nu_k := \sum_i r_{k,i}^{n-2} \delta_{a_{k,i}}$, and $\nu := \sum_{k=1}^\infty \nu_k$. Then ν is a finite positive measure supported on E , since $\nu(\mathbb{R}^{n+1}) \leq \sum_{k=1}^\infty 2^{-k} < \infty$. For every $x \in Z$ and every k , there is an i such that $|x - a_{k,i}| < r_{k,i}$, from which we deduce $U_{n-2}^{\nu_k}(x) \geq |x - a_{k,i}|^{2-n} r_{k,i}^{n-2} \geq 1$. It follows that

$$U_{n-2}^\nu(x) = \sum_{k=1}^\infty U_{n-2}^{\nu_k}(x) = +\infty \quad \text{for every } x \in Z.$$

Consequently, the finite positive measure $\mu := \lambda + \nu$ is supported on E and satisfies $U_{n-2}^\mu(x) = +\infty$ for every $x \in E$. Define now $\psi(x) := -U_{n-2}^\mu(x)$ so that $\psi = -\infty$ on E . If $x \notin E$, then $\mathrm{dist}(x, E) > 0$,

and hence $U_{n-2}^\mu(x) < \infty$. Thus $\{\psi = -\infty\} = E$. Moreover, on every compact subset of $\mathbb{R}^{n+1} \setminus E$, the kernel and all of its derivatives are uniformly bounded for $y \in E$; differentiation under the integral therefore gives $\psi \in C^\infty(\mathbb{R}^{n+1} \setminus E)$.

It remains to verify the Hessian inequality. For $z \neq 0$, writing $e = z/|z|$, one computes

$$D^2(-|z|^{2-n}) = (n-2)|z|^{-n}(I - ne \otimes e).$$

Hence, for every $P \in G(n, n+1)$, we have $\text{tr}_P D^2(-|z|^{2-n}) = (n-2)|z|^{-n}(n - n|\pi_P e|^2) = n(n-2)|z|^{-n}(1 - |\pi_P e|^2) \geq 0$. It follows that, for $x \notin E$,

$$\text{tr}_P D^2 \psi(x) = \frac{n}{\omega_n} \int_E \frac{1 - |\pi_P e_{x-y}|^2}{|x-y|^n} d\mu(y) \geq 0,$$

where $e_{x-y} := (x-y)/|x-y|$.

If $n = 2$, then $\mathcal{H}^0(E) < \infty$, so that $E = \{a_1, \dots, a_N\}$ is finite. In this case take

$$\mu := \sum_{j=1}^N \delta_{a_j}, \quad \text{and} \quad \psi(x) := \int K(|x-y|) d\mu(y) = \frac{1}{2\pi} \sum_{j=1}^N \log |x - a_j|.$$

Clearly $\{\psi = -\infty\} = E$ and $\psi \in C^\infty(\mathbb{R}^3 \setminus E)$. Since $D^2 \log |z| = |z|^{-2}(I - 2e \otimes e)$, for every $P \in G(2, 3)$, $\text{tr}_P D^2 \log |z| = 2|z|^{-2}(1 - |\pi_P e|^2) \geq 0$. $\square$

Remark B.3. The function ψ , extended by $-\infty$ on E , is n -plurisubharmonic in the sense of [HL14, §9]. Consequently the set E is C^∞ n -pluripolar (again, see [HL14, §9]).

Let us now adapt the basic construction of Lemma B.1 to the setting of Riemannian manifolds (with and without boundary). For $r > 0$, let

$$B_r := B_r^{n+1}(0), \quad C_r := B_r^n(0) \times [0, r), \quad \partial_0 C_r := B_r^n(0) \times \{0\} \subset C_r.$$

Lemma B.4. *Let $n \geq 2$.*

- (a) *Let g be a smooth Riemannian metric on a neighbourhood of $\overline{B}_2$. There exists $r_0 \in (0, 1)$ such that for every compact set $E \subset \overline{B}_{r_0}$ with $\mathcal{H}^{n-2}(E) < \infty$ there is a function $v \in C^\infty(B_{2r_0} \setminus E)$, $v \geq 1$ such that $v(x) \rightarrow +\infty$ as $x \rightarrow E$ and $\text{tr}_P \nabla_g^2 v(x) \leq 0$ for all $x \in B_{2r_0} \setminus E$ and every $P \in G(n, n+1)$.*
- (b) *Let g be a smooth Riemannian metric on a neighbourhood of $\overline{C}_2$ of the form $g = dt^2 + h_{ij}(y, t) dy^i dy^j$, where (h_{ij}) is smooth and positive-definite for all $t \in [0, 2]$. There exists $r_0 \in (0, 1)$ such that for every compact set $E \subset \overline{B}_{r_0}^n \times \{0\}$ with $\mathcal{H}^{n-2}(E) < \infty$ there is a function $v \in C^\infty(C_{2r_0} \setminus E)$, $v \geq 1$ such that $v(x) \rightarrow +\infty$ as $x \rightarrow E$ and $\text{tr}_P \nabla_g^2 v(x) \leq 0$ for all $x \in C_{2r_0} \setminus E$ and every $P \in G(n, n+1)$, and $\partial_t v = 0$ on $\partial_0 C_{2r_0} \setminus E$.*

In both cases, the extension of v by $+\infty$ on E is lower semicontinuous on B_{2r_0} and C_{2r_0} , respectively.

Proof. In order to deal with the errors arising from the Riemannian metric g , it is convenient to introduce a strictly superharmonic (with respect to the Euclidean metric) perturbation of the Riesz kernel in Lemma B.1. For $\lambda > 0$, choose $r_* > 0$ such that $\lambda r_* < n$, and define

$$(B.1) \quad K_\lambda(r) := \int_r^{r_*} s^{1-n} e^{\lambda s} ds \quad \text{for } 0 < r < r_*,$$

and notice that, as $r \downarrow 0$, K_λ has the same leading order behaviour as the Riesz kernel in Lemma B.1.

Now, for (a), by shrinking r_0 , let us assume that B_{3r_0} is strongly g -convex and for $a \in B_{r_0}$ the distance $r_a(\cdot) = \text{dist}_g(\cdot, a)$ is smooth on $B_{2r_0} \setminus \{a\}$ and satisfies $|\nabla_g^2 r_a - r_a^{-1}(g - dr_a \otimes dr_a)| \leq C r_a$

(which is a consequence of the Hessian comparison theorem in [Lee18, Theorem 11.7]). Then, if $P \in G(n, n+1)$ and $\alpha = |\pi_P^\mathbf{g} \nabla^\mathbf{g} r_a|_\mathbf{g}^2 \in [0, 1]$, we have

$$(B.2) \quad \text{tr}_P \nabla_\mathbf{g}^2 K_\lambda(r_a) \leq K_\lambda''(r_a) + \frac{n-1}{r_a} K_\lambda'(r_a) + Cr_a |K_\lambda'(r_a)| \leq (-\lambda + Cr_a) |K_\lambda'(r_a)|.$$

Take $\lambda = 1$ and reduce r_0 so that $Cr_a < 1$, then $\text{tr}_P \nabla_\mathbf{g}^2 K_1(r_a) \leq 0$ for all $P \in G(n, n+1)$.

If we let μ be the finite positive measure given in the proof of Lemma B.1, and

$$v(x) := 1 + \int_E K_1(\text{dist}_\mathbf{g}(x, y)) d\mu(y),$$

then it is easy to check that, after shrinking r_0 if needed, v has all the required properties, since $\text{dist}_\mathbf{g}$ is uniformly equivalent to the Euclidean distance on B_{3r_0} with constants going to 1 as $r_0 \rightarrow 0$. In particular, the integral diverges on E . Its extension by $+\infty$ on E is lower semicontinuous by Fatou's lemma, which gives $v(x) \rightarrow +\infty$ as $x \rightarrow E$.

For (b), write $h_t := h_{ij}(y, t) dy^i dy^j$ on C_2 and let $\hat{\mathbf{g}} := dt^2 + h_0(y)$ on the doubled cylinder $\hat{C}_2 := B_2^n(0) \times (-2, 2)$. For $a = (a', 0) \in E$, let $\hat{r}_a(y, t) := \text{dist}_{\hat{\mathbf{g}}}((y, t), (a', 0)) = (\text{dist}_{h_0}(y, a')^2 + t^2)^{1/2}$. Then notice that $\partial_t \hat{r}_a(y, 0) = 0$ for $y \neq a'$.

The interior computation in (B.2) implies that for every $P \in G(n, n+1)$, we have $\text{tr}_P \nabla_\mathbf{g}^2 K_\lambda(\hat{r}_a) \leq (-\lambda - C\hat{r}_a) |K_\lambda'(\hat{r}_a)|$. By smoothness of h_t in t we have $|h_t - h_0| = O(|t|)$, $|\mathbf{g} - \hat{\mathbf{g}}|_\mathbf{g} \leq C|t|$, $|\nabla^\mathbf{g} - \nabla^{\hat{\mathbf{g}}}|_\mathbf{g} \leq C$. Then,

$$\text{tr}_P \nabla_\mathbf{g}^2 K_\lambda(\hat{r}_a) \leq \text{tr}_P \nabla_{\hat{\mathbf{g}}}^2 K_\lambda(\hat{r}_a) + C(|t| |\nabla_{\hat{\mathbf{g}}}^2 K_\lambda(\hat{r}_a)|_{\hat{\mathbf{g}}} + |K_\lambda'(\hat{r}_a)|)$$

If $\lambda \hat{r}_a \leq n/2$, then $|\nabla_{\hat{\mathbf{g}}}^2 K_\lambda(\hat{r}_a)|_{\hat{\mathbf{g}}} \leq C\hat{r}_a^{-1} |K_\lambda'(\hat{r}_a)|$, so that using $|t| \leq \hat{r}_a$

$$(B.3) \quad \text{tr}_P \nabla_\mathbf{g}^2 K_\lambda(\hat{r}_a) \leq (-\lambda + C_0 + C_1 \hat{r}_a) |K_\lambda'(\hat{r}_a)|.$$

Choose first $\lambda > 2C_0 + 1$ and then $r_0 < r_*$ so small that $\lambda r_* < n/2$, and $C_1 \hat{r}_a \leq 1$. Then $\text{tr}_P \nabla_\mathbf{g}^2 K_\lambda(\hat{r}_a) \leq 0$ for all $P \in G(n, n+1)$.

Finally, if we define

$$v(x) := 1 + \int_E K_1(\text{dist}_\mathbf{g}(x, y)) d\mu(y),$$

for the finite positive measure μ in the proof of Lemma B.1, then it is easy to check that it satisfies all the required properties. Divergence of the integral at E follows as in (a), and $\partial_t v = 0$ on $\partial_0 C_{2r_0} \setminus E$ follows from $\partial_t \hat{r}_a(y, 0) = 0$. $\square$

We can now state and prove the main result of this appendix.

Proposition B.5. *Let $n \geq 2$, and let $(M^{n+1}, \mathbf{g})$ be a smooth Riemannian manifold-with-boundary with (possibly empty) boundary ∂M . Let $\mathcal{S} \subset M$ be a relatively closed subset with locally finite $\mathcal{H}^{n-2}$ measure, i.e. $\mathcal{H}^{n-2}(\mathcal{S} \cap K) < +\infty$ for every $K \Subset M$. Let V be an n -dimensional rectifiable varifold in M such that*

$$\delta V(X) = 0 \quad \text{for every } X \in \mathfrak{X}^{\text{tan}}(M \setminus \mathcal{S}).$$

Then V is free-boundary stationary in M , that is

$$\delta V(X) = 0 \quad \text{for every } X \in \mathfrak{X}^{\text{tan}}(M).$$

Consequently, if Σ is a smooth proper minimal free-boundary immersion in $M \setminus \mathcal{S}$ and $|\Sigma|$ has locally finite mass, then $|\Sigma|$ is free-boundary stationary in M .

Proof. Let $U \Subset M$ be a subset of the domain of a coordinate chart such that U is either a metric ball or a metric half-cylinder in Fermi coordinates with respect to ∂M as in Lemma B.4, and let $X \in \mathfrak{X}^{\text{tan}}(U)$. The global statement then follows from this local one after covering $\text{spt } X$ by finitely many sufficiently small metric balls or half-cylinders.

Choose $\text{spt } X \Subset U_0 \Subset U_1 \Subset U$ and set $E = \mathcal{S} \cap \overline{U}_1$, so that E is compact and satisfies $\mathcal{H}^{n-2}(E) < \infty$ by hypothesis. By Lemma B.4, there is a function $v \in C^\infty(U \setminus E)$ such that $v \geq 1$ on $\overline{U}_1 \setminus E$, $v(x) \rightarrow +\infty$ as $x \rightarrow E$ and $\text{tr}_P \nabla_g^2 v \leq 0$ for all $P \in G(n, n+1)$. In particular, v is superharmonic.

We now prove a logarithmic Caccioppoli estimate. Let $\eta \in C_c^\infty(U_1)$ satisfy $\eta \equiv 1$ on U_0 , and let

$$q_\ell(v) := \left(\frac{1}{v} - \frac{1}{\ell} \right)_+, \quad \text{for } \ell > 1.$$

Extend v by $+\infty$ on E . Lower semicontinuity and the fact that $\text{spt } \eta$ is compact imply that $\text{spt } \eta \cap \{v \leq \ell\}$ is a compact subset of $U_1 \setminus \mathcal{S}$. Thus $Y_\ell := \eta^2 q_\ell(v) \nabla^g v$ is supported away from $\mathcal{S}$ and is tangent to ∂M in the boundary case. Free-boundary stationarity away from $\mathcal{S}$ gives

$$0 = \delta V(Y_\ell) = \int_{G_n(U_1)} [\eta^2 q_\ell(v) \text{tr}_P \nabla_g^2 v + \langle \nabla_P(\eta^2 q_\ell(v)), \nabla_P v \rangle] dV(x, P).$$

The first term is non-positive, thus implying

$$\int_{G_n(U_1)} \langle \nabla_P(\eta^2 q_\ell(v)), \nabla_P v \rangle dV(x, P) \geq 0.$$

Now, on $\{v < \ell\}$, there holds $\nabla_P q_\ell(v) = -v^{-2} \nabla_P v$. Therefore,

$$\begin{aligned} A_\ell &:= \int_{\{v < \ell\}} \eta^2 |\nabla_P \log(v)|^2 dV \leq 2 \int_{\{v < \ell\}} \eta \left(1 - \frac{v}{\ell} \right) \langle \nabla_P \eta, \nabla_P \log(v) \rangle dV \\ &\leq \frac{1}{2} A_\ell + 2 \int |\nabla_P \eta|^2 dV. \end{aligned}$$

Rearranging,

$$A_\ell \leq 4 \int |\nabla_P \eta|^2 dV \leq 4 \|D\eta\|_\infty^2 \|V\|(\text{spt } \eta) < +\infty.$$

The last quantity is finite because V has locally finite mass by assumption. Letting $\ell \rightarrow \infty$ gives

$$\int_{U_0} |\nabla_P \log(v)|^2 dV < +\infty.$$

We now construct the cut-off. Choose $\vartheta \in C^\infty(\mathbb{R})$ such that $0 \leq \vartheta \leq 1$, $\vartheta \equiv 1$ on $(-\infty, 1]$ and $\vartheta \equiv 0$ on $[2, \infty)$. Define then $\chi_j := \vartheta(j^{-1} \log(v))$. For every fixed j , the function χ_j vanishes in an ambient neighbourhood of E , so that $\chi_j X$ is an admissible test vector field supported away from $\mathcal{S}$. Moreover, $\chi_j \rightarrow 1$ pointwise on $U_0 \setminus \mathcal{S}$. Lastly,

$$\int_{U_0} |\nabla_P \chi_j|^2 dV \leq \frac{\|\vartheta'\|_\infty^2}{j^2} \int_{U_0} |\nabla_P \log(v)|^2 dV \rightarrow 0,$$

as $j \rightarrow \infty$. Consequently, from the stationarity condition

$$0 = \delta V(\chi_j X) = \int \chi_j \text{div}_P X dV + \int \langle \nabla_P \chi_j, X \rangle dV,$$

we can use dominated convergence to pass the first term on the right-hand side to the limit, while for the second term we have

$$\left| \int \langle \nabla_P \chi_j, X \rangle dV \right| \leq \|X\|_\infty \|V\|(\text{spt } X)^{1/2} \left(\int_{\text{spt } X} |\nabla_P \chi_j|^2 dV \right)^{1/2} \rightarrow 0,$$

as $j \rightarrow \infty$. Thus, V satisfies $\delta V(X) = 0$ in U meaning that it is free-boundary stationary across $\mathcal{S}$. $\square$

We record the following consequence of Proposition B.5 which will be the form we shall use for excision arguments (see 2.5 for the notational convention in the integrals).

Corollary B.6. *Let $\mathcal{S}$ be as in Definition 2.6, then for every compact $K \subset M$ there exists a sequence of smooth cutoff functions $\{\chi_j\}_j$ on M , each supported away from $\mathcal{S}$, such that $0 \leq \chi_j \leq 1$, $\chi_j \rightarrow 1$ on $\Sigma \cap K$, and $\int_{\Sigma \cap K} |\nabla_\Sigma \chi_j|^2 \rightarrow 0$. This will sometimes be denoted by $\text{cap}_2(\mathcal{S}) = 0$ and referred to as having vanishing relative 2-capacity in Σ (see below).*

Remark B.7. A weaker hypothesis suffices for extending stationarity when a local area-growth bound is already available. Namely, assume $\mathcal{H}^{n-1}(\mathcal{S}) = 0$ and $\mathcal{H}^n(\Sigma \cap B_r(c)) \leq Cr^n$ for $x \in \mathcal{S}$ and all sufficiently small radii for some constant C uniform on compact sets. Then stationarity away from $\mathcal{S}$ extends across $\mathcal{S}$.

Indeed, fix $X \in \mathfrak{X}^{\text{tan}}(U)$ and choose a compact $K \Subset U$ whose relative interior contains $\text{spt } X$. Since $\mathcal{H}^{n-1}(\mathcal{S} \cap K) = 0$, for every $\varepsilon > 0$ we may cover $\mathcal{S} \cap K$ by finitely many balls $B_{r_i}(x_i)$, with $r_i < \varepsilon$ and $\sum_i r_i^{n-1} < \varepsilon$. Choose $0 \leq \eta \leq 1$, equal to 0 near $\bigcup_i B_{r_i}(x_i)$ and to 1 outside $\bigcup_i B_{2r_i}(x_i)$, with $|D\eta| \leq C \sum_i r_i^{-1} \mathbf{1}_{B_{2r_i}(x_i)}$. The Euclidean growth bound $\|V\|(B_r(x)) \leq Cr^n$ at $x \in \mathcal{S}$ gives

$$\int |\nabla_P \eta| dV \leq C \sum_i r_i^{n-1} \leq C\varepsilon, \quad \|V\|(\{\eta \neq 1\}) \leq C \sum_i r_i^n \leq C\varepsilon^2.$$

Since ηX is supported away from $\mathcal{S}$, $0 = \delta V(\eta X)$; hence

$$|\delta V(X)| \leq \|DX\|_\infty \|V\|(\{\eta \neq 1\}) + \|X\|_\infty \int |\nabla_P \eta| dV \leq C(\|DX\|_\infty \varepsilon^2 + \|X\|_\infty \varepsilon).$$

Letting $\varepsilon \downarrow 0$ proves that $\delta V(X) = 0$.

APPENDIX C. TANGENT CONES AND STRATIFICATION FOR FREE-BOUNDARY STATIONARY VARIFOLDS

C.1. Free-boundary stationary varifolds. An integral¹³ varifold V on M is said to be *free-boundary stationary* if its first variation vanishes for all compactly supported vector fields on M which are tangential to ∂M along $\partial M \subset M$, i.e.

$$(C.1) \quad \delta V(X) = 0 \quad \text{for all } X \in \mathfrak{X}^{\text{tan}}(M).$$

In particular, nothing is assumed on the first variation of V in the direction of vector fields that are not tangential to ∂M . Notice that by Definition 2.6, it is easy to see that if Σ is a singular properly embedded (or properly immersed) free-boundary minimal hypersurface, then the multiplicity-one varifold $|\Sigma|$ induced by it is a free-boundary stationary varifold.

Remark C.1. Note that if Γ is a boundary component of M , then the varifold $|\Gamma|$ is free-boundary stationary, independently of whether the mean curvature of Γ in M vanishes, even though Γ is a smooth hypersurface. On the other hand, notice that Γ would not be a free-boundary minimal hypersurface according to our Definition 2.2. Therefore, free-boundary stationary varifolds are strictly more general than free-boundary minimal hypersurfaces, even if we restrict to smooth objects.

¹³It is actually not necessary to assume integrality, but, for the purposes of this paper, we shall not need to consider any larger classes of varifolds.

Although it is not apparent from the definition, free-boundary stationary integral varifolds have locally bounded first variation by [Ede20, Proposition 3.2]¹⁴. From this, Edelen manages to deduce a compactness theorem for sequences of free-boundary stationary integral varifolds with locally uniformly bounded mass [Ede20, Theorem 3.5] from Allard's Compactness theorem [All72, 6.4]. In our simplified setting of singular free-boundary minimal hypersurfaces, the full first variation can be computed rather directly as in Lemma 5.4, so compactness can be reduced immediately to Allard's result.

If we consider free-boundary stationary varifolds in a Euclidean half-space, then we have the following rather useful reflection principle.

Lemma C.2 (Reflection principle, [LZ21b, Lemma 2.2]). *Let $v \in \mathbb{R}^{n+1}$ be a unit vector and let P^+ be the closed half-space $\{u : u \cdot v \geq 0\}$. Let $\theta : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ be the reflection map across $T = v^\perp$ given by $\theta(u) = u - 2(u \cdot v)v$. If V is a free-boundary stationary integral varifold in P^+ , then the doubled varifold*

$$\bar{V} := V + \theta_\# V$$

is a stationary integral varifold in $\mathbb{R}^{n+1}$ and it is invariant under θ , i.e. $\theta_\# \bar{V} = \bar{V}$.

Now let V be a free-boundary stationary integral varifold and let $x \in M$.

- If $x \in M \setminus \partial M$, then clearly for any open set $U \Subset M \setminus \partial M$ containing x , V is stationary in U and therefore the usual *interior monotonicity formula* (see [Sim83, §17]) holds for balls $B_r(x) \subset U$.
- If $x \in \partial M$, then [GLZ20, Theorem 3.4] provides a *boundary monotonicity formula* for balls centred at x .

The combination of these two monotonicity formulas implies existence of density and of tangent varifolds at every point. It also allows to show that any tangent varifold to V at a point $x \in \text{spt } \|V\|$ is a cone. (See Appendix C.2 for the exact definition in the manifold setting and some properties.)

On the other hand, some standard properties of stationary integral varifolds do not hold directly for free-boundary stationary integral varifolds. As an example, upper semicontinuity of density does not hold, even in the smooth setting (think of a sequence of *interior* points converging to a *boundary* point). In order to have a quantity that does not distinguish interior balls from balls centred at the boundary and satisfies a uniform monotonicity formula, it is possible to introduce the so-called *reflected density* (see [GJ86a, Section 3] for the Euclidean case) which turns out to be upper-semicontinuous. As the adaptation to the manifold setting is somewhat involved, we shall avoid introducing the reflected density, and work with the two distinct monotonicity formulas above, which will suffice for our purposes.

C.2. Tangent cones. Let V be a free-boundary stationary integral varifold in M , let $x \in \text{spt } \|V\|$, and $r_i \rightarrow 0$ be a sequence of positive numbers. We shall consider two cases.

- (1) If $x \in M \setminus \partial M$, let $0 < \rho_x < \min\{\text{inj}_x(M), \text{dist}_g(x, \partial M)\}$ so that the exponential map of M at x is a diffeomorphism of $B_{\rho_x}^{T_x M}(0) \subset T_x M$ onto its image $B_{\rho_x}^M(x)$, and let $\Psi_{x, r_i} : B_{r_i^{-1}\rho_x}^{T_x M}(0) \rightarrow M$ be given by

$$\Psi_{x, r_i}(v) := \exp_x(r_i v).$$

Let $g_{x, r_i} := r_i^{-2} \Psi_{x, r_i}^* g$. Then, notice that

$$(B_{r_i^{-1}\rho_x}^{T_x M}(0), g_{x, r_i}) \rightarrow (T_x M, g_{\text{Euc}}) \quad \text{locally smoothly.}$$

¹⁴This is true even without assuming integrality by [DM21].

- (2) If $x \in \partial M$, let $0 < \rho_x < \min\{\text{inj}_x(\partial M), \text{inj}_x(M)\}$ be such that the exponential map of ∂M at x is a diffeomorphism of $B_{\rho_x}^{T_x(\partial M)}(0) \subset T_x(\partial M)$ onto its image $B_{\rho_x}^{\partial M}(x) \subset \partial M$ and the normal exponential map $F_{\partial M}$ of ∂M is a diffeomorphism of $B_{\rho_x}^{\partial M}(x) \times [0, \rho_x)$ onto its image. Then let $\Psi_{x,r_i} : B_{r_i^{-1}\rho_x}^{T_x^+ M}(0) = T_x^+ M \cap B_{r_i^{-1}\rho_x}^{T_x M}(0) \rightarrow M$ be given by

$$(C.2) \quad \Psi_{x,r_i}(v + t\mathbf{n}(x)) := F_{\partial M}(\exp_x^{\partial M}(r_i v), r_i t),$$

for $v \in B_{r_i^{-1}\rho_x}^{T_x(\partial M)}(0)$, $t \geq 0$, $v + t\mathbf{n}(x) \in B_{r_i^{-1}\rho_x}^{T_x^+ M}(0)$. Again, let $g_{x,r_i} := r_i^{-2}\Psi_{x,r_i}^*g$, and notice that

$$(B_{r_i^{-1}\rho_x}^{T_x^+ M}(0), \partial_0 B_{r_i^{-1}\rho_x}^{T_x^+ M}(0), g_{x,r_i}) \rightarrow (T_x^+ M, T_x(\partial M), g_{\text{Euc}}) \quad \text{locally smoothly.}$$

In either case, let Φ_{x,r_i} denote the inverse map Ψ_{x,r_i}^{-1} .

If we apply Edelen's Compactness theorem [Ede20, Theorem 3.5] to the sequence $V_i := (\Phi_{x,r_i})_{\#} V^{15}$, then, up to a subsequence (not relabelled), there exists an integral varifold V_{∞} such that $V_i \rightarrow V_{\infty}$ locally in the sense of varifolds. By convergence of the first variation, we can pass the variational property of V to the limit. More precisely, we have the following.

- (1) If $x \in M \setminus \partial M$, then V_{∞} is a stationary integral varifold in $(T_x M, g_{\text{Euc}}) \cong (\mathbb{R}^{n+1}, g_{\text{Euc}})$. The usual monotonicity formula for stationary varifolds applies and one deduces that V_{∞} is a stationary integral cone of density $\Theta(\|V\|, x)$.
- (2) If $x \in \partial M$, then V_{∞} is a free-boundary stationary integral varifold in $(T_x^+ M, g_{\text{Euc}}) \cong (\mathbb{R}_+^{n+1}, g_{\text{Euc}})$. The monotonicity formula for free-boundary stationary varifolds of [GLZ20] applies, and one deduces that V_{∞} is a free-boundary stationary integral cone of density $\Theta(\|V\|, x)$. Moreover, if we denote by θ_0 the map on $(T_x M, g_{\text{Euc}})$ given by reflection in $T_x(\partial M) \subset T_x M$, then, by the reflection principle in [LZ21b, Lemma 2.2], the *doubled* cone

$$(C.3) \quad \bar{V}_{\infty} := V_{\infty} + (\theta_0)_{\#} V_{\infty}$$

is a stationary integral cone in $(T_x M, g_{\text{Euc}})$, which is also invariant under θ_0 .

Any varifold V_{∞} that arises in this way is said to be a *tangent cone* to V at x . The set of all tangent cones to V at x is denoted by $\text{VarTan}(V, x)$.

C.3. Singular set and stratification for free-boundary stationary integral varifolds. In order to analyse the regularity of a free-boundary stationary integral varifold V , we need to introduce some further definitions.

Definition C.3 (Regular and singular set). Let V be a free-boundary stationary integral varifold in M . We define the *regular set* $\text{reg}(V)$ of V to be

$$\text{reg}(V) := \left\{ x \in \text{spt } \|V\| : \begin{array}{l} \text{there exists } r > 0 \text{ such that } \text{spt } \|V\| \cap B_r(x) \text{ is a smooth} \\ \text{embedded hypersurface-with-(possibly empty) boundary in } \partial M \end{array} \right\}.$$

Similarly, we define the *singular set* $\text{sing}(V)$ of V by

$$(C.4) \quad \text{sing}(V) = \text{spt } \|V\| \setminus \text{reg}(V).$$

We shall now subdivide the singular set into pieces according to the possible tangent cones that can occur. In order to state the results of this stratification procedure, we need some more definitions and notation about cones. Let $\eta_{v,\lambda}$ be the map $x \mapsto \lambda^{-1}(x - v)$.

¹⁵It is easy to see that [Ede20, Theorem 3.5] applies to a sequence of metrics converging in C_{loc}^2 .

Definition C.4 (Spine of a cone). Let $\mathbf{C}$ be an n -dimensional cone in $\mathbb{R}^{n+1}$. We define the *spine* $S(\mathbf{C})$ of $\mathbf{C}$ to be the translation invariance set of $\mathbf{C}$, namely

$$(C.5) \quad S(\mathbf{C}) := \{v \in \mathbb{R}^{n+1} : (\eta_{v,1})_{\#} \mathbf{C} = \mathbf{C}\},$$

which is a vector subspace of $\mathbb{R}^{n+1}$.

For $x \in \text{spt } \|V\|$ and $\mathbf{C} \in \text{VarTan}(V, x)$ let

$$(C.6) \quad \widehat{\mathbf{C}} = \begin{cases} \mathbf{C} & \text{if } x \in M \setminus \partial M, \\ \mathbf{C} + (\theta_0)_{\#} \mathbf{C} & \text{if } x \in \partial M. \end{cases}$$

Definition C.5 (j -stratum). For $j \in \{0, 1, \dots, n\}$, we define the j -stratum $\mathcal{S}_j$ as follows:

$$(C.7) \quad \mathcal{S}_j := \{x \in \text{sing}(V) : \dim(S(\widehat{\mathbf{C}})) \leq j \text{ for all } \mathbf{C} \in \text{VarTan}(V, x)\}$$

Remark C.6. Clearly,

$$(C.8) \quad \mathcal{S}_0 \subset \mathcal{S}_1 \subset \dots \subset \mathcal{S}_n = \text{sing}(V).$$

The usefulness of this stratification of the singular set is due to the following result, which was first proved by Almgren in the setting of area-minimising currents.

Proposition C.7 ([LPS26, Proposition 3.30]). *Let V be a free-boundary stationary integral varifold in M . Then, for all $j \in \{0, 1, \dots, n\}$*

$$(C.9) \quad \dim_{\mathcal{H}} \mathcal{S}_j \leq j,$$

where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension.

APPENDIX D. NON-EMPTYNESS OF THE BOUNDARY

We expand on Remark 1.10. In particular, we first provide an example of a compact Riemannian manifold-with-boundary (M, g) where the one-parameter min-max construction of Theorem 1.8 yields a stationary integral varifold V which is supported on an optimally regular minimal hypersurface entirely contained in the interior of M , then show that this cannot happen under the additional assumption that $\text{Ric}_g \geq 0$ on M and ∂M is strictly mean-convex.

D.1. $\text{spt } \|V\| \cap \partial M = \emptyset$. For sufficiently large $L > 0$, let the manifold M_L be the following truncated ellipsoid

$$E_L = \left\{ (x, z) \in \mathbb{R}^{n+1} \times \mathbb{R} : |x|^2 + \frac{z^2}{L^2} = 1 \right\} \subset \mathbb{R}^{n+2}, \quad M_L = E_L \cap \{z \leq L/2\},$$

endowed with the induced metric coming from the Euclidean metric on $\mathbb{R}^{n+2}$. The latitude sweepout $t \mapsto M_L \cap \{z = L/2 - t\}$ for $t \in [0, 3L/2]$ implies that the first free boundary min-max width $W_1(M_L, \partial M_L)$ of M_L is at most $\mathcal{H}^n(\mathbb{S}^n)$, independently of L .

For $t \in [0, L/2]$, the level sets $M_L \cap \{z = L/2 - t\}$ have mean curvature vector pointing towards ∂M_L . In particular, by White's varifold maximum principle [Whi10], if Σ is a singular properly immersed free-boundary minimal hypersurface with $\overline{\Sigma} \cap \partial M_L \neq \emptyset$, then $\overline{\Sigma}$ must intersect every level set $M_L \cap \{z = t\}$ for $t \in [0, L/2]$. On the other hand, by applying interior monotonicity to $\Sigma \cap \{L/8 \leq z \leq 3L/8\}$, we obtain $\mathcal{H}^n(\Sigma) \geq c_n L$ whenever $\overline{\Sigma} \cap \partial M_L \neq \emptyset$, where $c_n > 0$ is independent of L . Consequently, for L sufficiently large, every singular free-boundary minimal hypersurface realising $W_1(M_L, \partial M_L)$ must be disjoint from ∂M_L .

D.2. $\text{Ric}_g \geq 0$ on M and strictly mean-convex boundary. Let us then give a quick proof of the following simple fact.

Lemma D.1. *If (M, g) is a compact connected Riemannian manifold-with-boundary with $\partial M \neq \emptyset$ such that $\text{Ric}_g \geq 0$ and ∂M is strictly mean-convex (namely $H_{\partial M} > 0$), and the free boundary stationary integral varifold V in M is induced by a non-empty singular properly immersed free-boundary minimal hypersurface Σ , then $\partial \Sigma \neq \emptyset$.*

Proof. We shall argue by contradiction. Notice that $\bar{\Sigma} = \text{spt } \|V\|$. By compactness of M , there is $c > 0$ such that $H_{\partial M} \geq c$; so that, by [CF18, Lemma 2.1–2.2], the distance function $d_{\partial M}$ from ∂M satisfies $\Delta_g d_{\partial M} \leq -c$ in the barrier sense¹⁶. Now suppose $x \in \bar{\Sigma}$ is such that $d_{\partial M}(x) = \text{dist}_g(\bar{\Sigma}, \partial M) > 0$. For $\varepsilon \in (0, c/2)$, the construction in the proof of [CF18, Lemma 2.2], applied to the strictly mean-convex hypersurface ∂M , gives a neighbourhood U of x in M and a smooth upper barrier function $u : U \rightarrow \mathbb{R}$ for $d_{\partial M}$ such that $|\nabla^g u| = 1$ on U and $\Delta_g u(x) \leq -c + \varepsilon \leq -c/2 < 0$. For $y \in U \cap \bar{\Sigma}$, we have

$$u(x) = d_{\partial M}(x) \leq d_{\partial M}(y) \leq u(y)$$

since u is an upper barrier for $d_{\partial M}$ and x minimises distance to ∂M . Therefore, $\bar{\Sigma} \cap U$ lies on one side of $S = \{u = u(x)\}$ and it touches S at x . Since $|\nabla_g u| = 1$, it is easy to check that $H_S(x) = -\Delta_g u(x) \geq c/2 > 0$, and the domain $\{u \geq u(x)\} \supset \bar{\Sigma} \cap U$ is strongly n -convex at x . The contradiction then follows from White’s varifold maximum principle [Whi10]. $\square$

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¹⁶This is stated clearly in Lemma 2.1 of [CF18] for the region where $d_{\partial M}$ is smooth, and it follows from the proof of Lemma 2.2 (in the barrier sense) outside this region. This statement also appears (without proof) as Proposition 5 in [Wan21]. A similar barrier construction appears in [Per16, Proof of Theorem 1.1].

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